



Exercise sheet

Algebraic expressions: Notable identities

Difficulty: ★☆☆☆ very easy ★★☆☆ easy ★★★☆ medium ★★★★★ hard

Track: 🏹 core 🎯 recommended 🏰 going further

Progressive reference notes: to be completed throughout the chapter.

Nature of an expression (part 1 of the course notes). The main operation acts on the whole expression. Expanding turns a product into a sum; factorising turns a sum into a product. A square reads $E^2 = E \times E$.

Prior knowledge from Year 9. $k(a + b) = ka + kb$ and $ka + kb = k(a + b)$.

Expanding double brackets (part 2): $(a + b)(c + d) = \dots\dots\dots$

Notable identities (part 3):

$$(a + b)^2 = \dots\dots\dots \quad (a - b)^2 = \dots\dots\dots \quad (a + b)(a - b) = \dots\dots\dots$$

Writing integers (parts 1 and 5). Even: $\dots\dots\dots$ Odd: $\dots\dots\dots$ Three consecutive integers: $\dots\dots\dots$

Pitfall log, to be completed throughout the chapter:

- A sign or a multiplication inside does not determine the nature: we spot $\dots\dots\dots$
- The opposite of a sum or a difference changes $\dots\dots\dots$
- The conjecture from the table (activity in the course notes) and what proved it: $\dots\dots\dots$
- When a coefficient is squared, what must be squared: $\dots\dots\dots$
- To factorise, the first structure to look for: $\dots\dots\dots$
- Logic: examples do not prove a general statement; a single counterexample is enough to disprove it.

Cumulative method, taking a snapshot of the structure:

1. Read. Which operation acts on the whole expression? What structure do we want to obtain?
2. Expand. Write all products with their signs before simplifying.
3. Recognise. Name the structure before applying a notable identity.
4. Factorise. Look first for a common factor, and only then for another structure.
5. Always check. Redo the transformation in reverse or substitute a simple value.

1. Nature of an expression and expanding

Quick warm-up (previous chapters)

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- a) Write $\frac{84}{126}$ in lowest terms (prime factorisation).
- b) Write in the form 2^n : $2^3 \times 2^5$; $\frac{2^9}{2^4}$.
- c) Give the standard form (scientific notation) of 0,0072.
- d) Calculate $\frac{3}{4} - \frac{1}{6}$ as a fraction in lowest terms.
- e) Two similar triangles have sides with a scale factor of 2: what can be said about the ratio of their areas?

1.1. Sum or product? Sorting

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For each expression, state whether it is a sum or a product (circle the main operation, the one acting on the whole expression), then name its terms or its factors.

- | | |
|---------------------|--------------------------|
| a) $3x + 2$ | f) $(2x - 3)^2$ |
| b) $5(x + 4)$ | g) $3(x + 1) + 2$ |
| c) $(x + 1)(x - 3)$ | h) $x(x + 1) + 3(x + 2)$ |
| d) $x^2 - 9$ | i) $2x^2$ |
| e) $7 - 5x$ | j) $(x + 5)(x - 5)$ |

1.2. Distributive property and opposite

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- a) Expand and simplify: $3(2x + 5)$; $-4x(6 - 3x)$; $7(3 - x) + 2x$.
- b) Give the opposite of each expression: $5 - 4x$; $3x - 7$; $-2x + 9$; $x^2 - 6x + 1$.
- c) Simplify: $5x + 3 - 2x + 8$; $4x^2 - x + 3x^2 + 7x - 9$.

1.3. Expanding double brackets: drills

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Expand and simplify.

- | | |
|-----------------------|---------------------------|
| a) $(x + 3)(x + 5)$ | e) $-4x(6 - 3x) - 8x^2$ |
| b) $(x - 3)(2x + 5)$ | f) $3(2x + 1) - (6 - x)$ |
| c) $(3x + 1)(2x - 7)$ | g) $(-3x + 5)(2x - 3)$ |
| d) $(2x - 3)(5x + 7)$ | h) $(x + 6)(x - 2) - x^2$ |

1.4. Mixed expressions to simplify

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Expand and simplify each expression.

- a) $A = 5(4x - 7) + (x + 9)(5x - 6)$
- b) $B = (x - 5)(x + 7) + (2x - 4)(3x - 1)$
- c) $C = 3x(x - 3) - (x + 7)(4x - 2)$
- d) $D = (y - 9)(3y + 2) - (2y - 7)(4y - 5)$

1.5. Proving the double distributive property

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We want to prove that, for all real numbers a, b, c, d : $(a+b)(c+d) = ac+ad+bc+bd$, using only the distributive property $k(c+d) = kc+kd$.

- Let $k = a + b$. Write $(a+b)(c+d)$ using k , and expand with the distributive property.
- In the result, replace k with $a + b$, and expand each of the two resulting products (distributive property, this time distributing from the right: $(a+b)c = ac+bc$).
- Put the four terms back together, then rearrange them in the order $ac+ad+bc+bd$, naming the property used. How many times was the distributive property applied in total?
- Going further: adapt the proof to obtain the formula for $(a+b)(c-d)$.

2. Notable identities: expanding

2.1. Drills

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Expand and simplify using the appropriate identity, writing the “formula” step before the result.

- | | | |
|---------------|-----------------|-----------------|
| a) $(x+3)^2$ | d) $(y-6)^2$ | g) $(t-5)(t+5)$ |
| b) $(x+10)^2$ | e) $(a-7)(a+7)$ | h) $(n+1)^2$ |
| c) $(x-4)^2$ | f) $(x+8)(x-8)$ | i) $(4-a)^2$ |

2.2. With coefficients

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Expand and simplify.

- | | |
|---------------|-------------------|
| a) $(2x+1)^2$ | e) $(3x-2)(3x+2)$ |
| b) $(3x+7)^2$ | f) $(5x+7)(5x-7)$ |
| c) $(3-2x)^2$ | g) $(y-2,5)^2$ |
| d) $(4x-3)^2$ | h) $(1-a)(1+a)$ |

2.3. In reverse: recognising an expanded form

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- a) Each of the following expressions is the expansion of a notable identity. Find the original expression:

$$x^2 + 10x + 25; \quad x^2 - 6x + 9; \quad 4x^2 - 12x + 9; \quad 4y^2 + 40y + 100; \quad 49t^2 - 14t + 1.$$

- b) Missing number. Find the missing number so that the expression is the square of a sum or of a difference:

$$x^2 + \dots x + 9; \quad 4x^2 - 4x + \dots; \quad \dots x^2 + 100x + 100.$$

- c) For the first expression in b), how many possible answers are there?

2.4. True or false?

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For each statement: state whether it is true or false, then justify. If it is true: a calculation for an equality between given numbers, an identity or an algebraic calculation for a “for all” statement, an example for a “there exists” statement. If it is false: a numerical counterexample.

- For all numbers a : $(2a+3)^2 = 4a^2 + 9$.
- The square of the difference of two numbers is equal to the difference of the squares of these two numbers.
- $26^2 - 25^2 = 26 + 25$.
- For all numbers a : $(a+5)^2 - (a-5)^2 = 20a$.
- For all numbers x , $x^2 - 12x + 9$ is the square of a number.
- For all numbers a and b : $(a+b)^2 = a^2 + b^2$.
- There exist numbers a and b for which $(a+b)^2 = a^2 + b^2$.

2.5. With fractions

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Expand and simplify.

a) $\left(\frac{1}{2} + x\right)\left(\frac{1}{2} - x\right)$

d) $\left(\frac{t}{3} + 6\right)^2$

b) $\left(\frac{x}{2} - 3\right)^2$

e) $\left(\frac{2x}{5} - 1\right)\left(\frac{2x}{5} + 1\right)$

c) $\left(\frac{1}{2} - 3x\right)^2$

2.6. Three forms, a single expression

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Consider the two expressions:

$$A = a^2 - ab + b^2 \quad \text{and} \quad B = (a + b)^2 - 3ab.$$

- a) Calculate A and B for $a = 5$ and $b = 3$. What do you notice?
- b) Without calculating anything, Zoé bets that for $a = 2\,023$ and $b = 1\,515$, the two expressions will still give the same result. At this stage, is this bet already proved?
- c) Prove that $A = B$ for all numbers a and b (expand B and simplify). Was Zoé's bet a sure thing from the start?
- d) Cultural remark (accepted result). When $a + b \neq 0$, the fraction $C = \frac{a^3 + b^3}{a + b}$ is also equal to A . Check this fact for $a = 5$ and $b = 3$.

2.7. The formula for $(a + b + c)^2$

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We want to prove that, for all numbers a , b , c :

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca).$$

- a) Test the formula for $a = 1$, $b = 2$, $c = 3$ (calculate both sides separately). Is this enough to conclude?
- b) Reuse the block trick (from the proof of the double distributive property, in 1.5): write $(a + b + c)^2 = [(a + b) + c]^2$, then apply the square of a sum identity to the two blocks $(a + b)$ and c .
- c) Complete the expansion, simplify, and conclude.
- d) Structure check: the expansion must contain three squares and three double products, one double product per pair of letters. Check that the result obtained contains all of them.

3. Factorising

3.1. Common factor

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Underline a common factor, then factorise by taking out all common factors still visible.

a) $5a + 15b$

c) $7x - 35$

e) $8 + 2x$

b) $12x^2 - 15x$

d) $x^2 - 5x$

f) $30a + 36a^2$

3.2. Compound common factor

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The common factor is a whole bracket. Factorise, then simplify the second bracket.

a) $A = (x - 2)(x + 3) + (x - 2)(x + 6)$

b) $B = (x + 5)(4x - 2) - (x + 5)(9x + 1)$

c) $C = (3x + 8)^2 - (6x - 7)(3x + 8)$

d) $D = (4x + 5)^2 - 5(4x + 5)$

3.3. Factorising with identities: pitfalls

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Recognise the structure (take a snapshot: how many terms? squares? the double product?), then factorise if possible using the tools from this chapter. In two cases, none of the three identities allows us to conclude here: which ones? For each, explain precisely what blocks us and state whether this obstacle is temporary or not within the scope of this chapter.

- a) $x^2 - 64$
- b) $49x^2 - 16$
- c) $x^2 + 2x + 1$
- d) $9y^2 - 24y + 16$
- e) $25 - 100t^2$
- f) $64 - 80x + 25x^2$
- g) $36z^2 - 1$
- h) $x^2 - 13$
- i) $16x^2 - 144$
- j) $x^2 + 81$
- k) $4x^2 - 49$
- l) $x^2 - 20x + 100$

3.4. Nested differences of squares

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Here, a or b is a bracketed block. Factorise, then simplify each bracket (and factor out a number if possible).

- a) $A = (x + 1)^2 - 36$
- b) $B = 25 - (2x + 3)^2$
- c) $C = 4x^2 - (x - 5)^2$
- d) $D = (8x - 4)^2 - 9x^2$

3.5. The two-way table

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Copy and complete the table (each row uses a notable identity, in one direction or the other).

Factorised form	Expanded form
$(x + 30)^2$	
	$x^2 - 12x + 36$
	$25y^2 + 50y + 25$
$(t - 8)(t + 8)$	
	$x^2 - 81$
$(7t - 5)^2$	

3.6. Louise’s work (spot the mistake)

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To factorise $E = 4x^2 - 4x + 1$, Louise handed in the following work:

“I can see the common factor $4x$, so $E = 4x(x - 1)$. It is factorised, I’m done!”

- a) Expand $4x(x - 1)$. Do we obtain E ? Which term of E disappeared in Louise’s work?
- b) Check in another way, without expanding: calculate E and then $4x(x - 1)$ for $x = 1$. Conclude.
- c) Give the correct factorisation of E , and check it by expanding.
- d) Explain the difference between the two previous checks: what does the test at $x = 1$ allow us to conclude? What does re-expanding $(2x - 1)^2$ establish?

3.7. Factorising by grouping terms

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Method (four or more terms). When there is neither a visible common factor nor a recognisable identity: group the terms in pairs, factorise each group, and bring out a common bracket. The choice of groups can change everything: if one grouping fails, try another.

- a) Factorise $A = ab + a + b + 1$ following the grouping $(ab + a) + (b + 1)$.
- b) We want to factorise $B = a^2 - a - x^2 - x$. First try the grouping $(a^2 - a) - (x^2 + x)$: factorise each group, and explain why we get stuck.
- c) Try B again with the grouping $(a^2 - x^2) - (a + x)$, and factorise completely.
- d) Check the result of c) by substituting $a = 3$ and $x = 1$ into B and into the product obtained.

4. Clever mental maths (the reward)

The principle. Rewrite without changing the value, then choose the route that truly reduces the remaining operations: notable identity or distributive property. A “round” number is a clue, not a rule. Examples: $61^2 = (60 + 1)^2$; $52 \times 48 = (50 + 2)(50 - 2)$; $62^2 - 58^2 = (62 + 58)(62 - 58)$. No written column multiplication is allowed in this section: that is the game.

4.1. Squares close to round numbers 🏹

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Calculate mentally, writing down the identity used, the calculation of each term and the result, then check.

- | | | | |
|------------|--------------------|-----------|-------------|
| a) 101^2 | c) 102×98 | e) 21^2 | g) 35^2 |
| b) 99^2 | d) 103×97 | f) 19^2 | h) 1001^2 |

4.2. In a single multiplication 🎯

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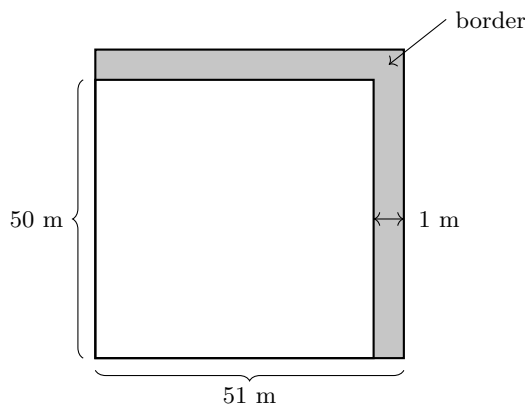
Calculate mentally, stating the identity used.

- | | | |
|-------------------|--------------------|--|
| a) $73^2 - 67^2$ | c) $43^2 - 42^2$ | e) $45^2 + 2 \times 45 \times 5 + 5^2$ |
| b) $101^2 - 99^2$ | d) $8,5^2 - 1,5^2$ | f) $14^2 - 2 \times 14 \times 4 + 4^2$ |

4.3. The area between two squares 🎯

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A square of side 51 m contains a square of side 50 m (figure below).



The figure is not to scale.

- Mentally calculate the area of the grey border (the surface between the two squares).
- Before validating: is this area plausible? The grey border is an L-shape made of two strips 1 m wide, along two adjacent sides only; estimate its area without exact calculation, then compare with the result of a).
- Generalise: for any real number $n > 0$, show that the area between a square of side $n + 1$ and a square of side n , placed as in the figure, is $2n + 1$. Where have we already met this expression in the chapter?

4.4. Champions: the big league 🎯

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Mentally, choosing and justifying the most economical transformation: notable identity or distributive property.

- | | |
|----------------------|-----------------------------|
| a) 1005^2 | c) 38×998 |
| b) $10,1 \times 9,9$ | d) $2026^2 - 2025^2 - 4051$ |

4.5. Checking without calculating

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The following statements are put forward as true. When a letter appears, the equality is claimed to hold for all real numbers. Disprove each statement in a single sentence, without full expansion or written calculation (structure, sign, units digit, test value...). In the items with a letter, a counterexample disproves “for all” without meaning that the equality fails for every value.

- a) “ $(x + 10)^2 = x^2 + 100$.”
- b) “ $25 - 100t^2 = (5 - 10t)^2$.”
- c) “ $47 \times 53 = 2\,500$.”
- d) “ $(3x)^2 = 3x^2$.”

4.6. Fixed perimeter, maximum area: the square wins

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Among all rectangles with perimeter 40 m, which one has the largest area?

- a) Check that the rectangles 12×8 , 15×5 and $10,5 \times 9,5$ do indeed have a perimeter of 40 m, then calculate their areas.
- b) A rectangle with perimeter 40 m has two dimensions whose sum is 20. After naming the larger one first, explain why they can be written $10 + d$ and $10 - d$ with $0 \leq d < 10$.
- c) Expand $(10 + d)(10 - d)$. Deduce that the area never exceeds 100 m^2 , and that it reaches 100 m^2 for only one shape: which one?
- d) Related problem (fixed area), new proof. A rectangle with area 100 m^2 has sides $x > 0$ and $y > 0$ such that $xy = 100$. Starting from $(x - y)^2 \geq 0$, show that its perimeter is at least 40 m and determine the case of equality.

Reference note “Justifying and reasoning”. True/false: 2.4; flawed work: 3.6; plausibility without calculation: 4.5; independent writing and proof: 5.1 to 5.4. Start by stating what the exercise asks for (disprove, correct, prove) before calculating.

5. Calculation programs and proofs

5.1. Arthur’s program: two trials are not enough to conclude for all x

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Consider the calculation program below.

Calculation program

- Choose a starting number.
- Multiply this number by (-2) .
- Add 5 to the product.
- Multiply the result by 5.
- Write down the result obtained.

- a) Check that when the starting number is 2, the result is 5.
- b) When the starting number is 3, what result do we obtain?
- c) What starting number must be chosen for the result to be 0?
- d) Arthur claims that, for any starting number x , the expression $(x - 5)^2 - x^2$ gives the result of the program. Can we decide by testing $x = 2$ and $x = 3$? Is he right? Prove it.

Modify, create, solve

1. Modify and predict. In Arthur’s program, replace the final multiplication by 5 with a multiplication by 3. Without recalculating everything, predict whether the starting number that gives 0 changes, then check algebraically.
2. Create under constraints. Invent a program of at least three steps whose result, for a starting number x , simplifies to exactly $6x - 12$. The instructions must be executable in a single unambiguous order.
3. Solve and check. Translate the program into an expression, prove that it equals $6x - 12$ and check it with two numbers.

5.2. Two programs, the same result?

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Program A

- Choose a number.
- Subtract 1.
- Square the result.
- Subtract 1.

Program B

- Choose a number.
- Subtract 2.
- Multiply by the chosen number.

- Apply both programs to the numbers 3, 10 and -5 . What do you notice? Make a conjecture.
- Prove the conjecture (express the result of each program for a starting number x).

5.3. Brevet-style: the squared program

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Consider the program: choose a starting number; add 1; square the result; subtract the square of the starting number from it; write down the final result.

- Check that when the starting number is 1, the result is 3.
- When the starting number is 2, what result do we obtain?
- With the starting number being x , express the final result in terms of x , then expand and simplify $P = (x + 1)^2 - x^2$. Use expanding double brackets here (a second method consists in factorising P as a difference of two squares).
- What starting number must be chosen to obtain 15?

5.4. Spreadsheet conjecture, algebraic proof: three consecutive integers

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We are interested in the sum of three consecutive integers (for example $7 + 8 + 9$).

- Calculate $(-8) + (-7) + (-6)$, then $7 + 8 + 9$, then $199 + 200 + 201$. Divide each result by 3. What do you notice?
- We prepare the spreadsheet below. Select **A2:A3** and extend this series down to **A201**. What formulas should be entered in **B2**, **C2** and **D2**, then copied down to row 201?

	A	B	C	D
1	1 st integer	2 nd integer	3 rd integer	sum
2	-100			
3	-99			

- State the conjecture. What exactly do the 200 displayed rows establish? Why are they not enough to prove a statement about all integers?
- Prove the conjecture. Choose a letter for one of the three numbers, specify which one it denotes, then write out the proof with an arbitrary integer, a chain of justified equalities and a “for all integers” conclusion.
- Show furthermore that the sum is always three times one of the three integers. Which one?
- Is the sum of four consecutive integers always a multiple of 4? Prove it, or disprove it with a counterexample.

5.5. The difference of the squares of two consecutive odd numbers

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Anabelle claims: “The difference of the squares of two consecutive odd numbers is always a multiple of 8.”

- Test the claim on $(3; 5)$, $(7; 9)$ and $(11; 13)$.
- Write two consecutive odd numbers using an integer n , then prove that Anabelle’s claim is true. Hint: factorising $a^2 - b^2$ is faster than expanding; compare with the model in the course notes.
- Where can the sum and the difference of the two starting numbers be found in the factorised result?

5.6. Divisible by 7 (echo of chapter 01)

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Let n be a positive integer. Prove that the number

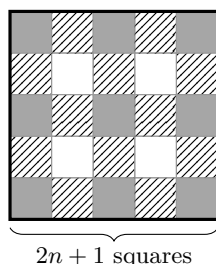
$$2^n + 2^{n+1} + 2^{n+2}$$

is always divisible by 7. Hint: a sum cannot be “combined”, it is factorised — a reminder from chapter 01 ($4^{15} + 8^{10}$): factor out 2^n .

5.7. The checked tablecloth 🍷

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On a tablecloth with white, red and striped squares, Zed observes a large square whose four corner squares are red: each side of this square then has an odd number of squares, written $2n + 1$ (with n an integer ≥ 1). The colours alternate as in the figure (side 5, i.e. $n = 2$; the red squares are shown in dark grey).



- In the figure, count the squares of each type. Check that the total is indeed 5^2 .
- In the square of side $2n + 1$, the “red-corner” rows alternate red/striped (they start and end with a red square); the other rows alternate striped/white. How many rows of each type are there? How many squares of each type does each row contain?
- Deduce, in terms of n , the number of red squares, the number of white squares and the number of striped squares (give factorised expressions).
- Check that the sum of the three expressions in c) equals $(2n + 1)^2$ (expand and simplify). Why was this check predictable?
- Numerical check: for $n = 2$, do we recover the counts from a)?

5.8. A decrease followed by an increase of the same percentage 🍷

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An investment with a strictly positive initial capital loses 50 % one year, then gains 50 % the following year. Many people think: “we are back to the starting point”. That is false.

- A portfolio is worth 1 000 €. It loses 50 %, then gains 50 %. How much is it worth at the end, and what percentage has it actually lost?
- Repeat with a loss then an increase of 10 %, then with 30 %. What does the number by which the initial capital is multiplied become each time?
- Prove the general case: losing a proportion x (with $0 < x < 1$) and then gaining the same proportion x amounts to multiplying the capital by $(1 - x)(1 + x)$. Expand this product, and deduce that the result always remains below the initial capital as soon as $x \neq 0$.
- Does the order change anything? Compare “gaining x then losing x ” and “losing x then gaining x ”. Justify.

Cross-cutting block: Justifying and reasoning 🍷

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- Spot the error. A pupil expands and simplifies $E = (2x + 3)^2 - (x - 1)(2x + 3)$:

$$E = 4x^2 + 9 - (2x^2 + 3x - 2x - 3) = 4x^2 + 9 - 2x^2 - x - 3 = 2x^2 - x + 6.$$

Locate precisely the first false equality, name the error made, then give the correct simplified form of E . Finally, check the result by factorising E by $(2x + 3)$.

- Check consistency without calculation. Each equality is claimed to be true for every real number x . Refute this claim in one line, using a counterexample or an identity. An equality may still be true for certain values of x .

- $(x + 5)^2 = x^2 + 25$
- $(5x)^2 = 5x^2$
- $x^2 - 64 = (x - 8)^2$
- $x^2 + 16 = (x + 4)^2$

- True or false? Justify each answer.

- “ $3x + 2$ is a product, since we can see the multiplication $3 \times x$ in it.”
- “Every sum can be factorised into a product of two first-degree expressions with real coefficients.”
- “The identities in this chapter do not give a general formula for writing $a^2 + b^2$ as a product of two first-degree expressions in a and b .”

- Write out a solution. Prove that, for every integer n , the number $(n + 1)^2 - n^2$ is odd. Your solution must include: the choice and status of the letter, the algebraic transformation, writing it in a form that proves it is odd, and a universal conclusion. Finish by checking with two values of n .

6. Brevet-style questions

This bank brings together exercises in the style of the brevet (French national exam at the end of Year 10), sorted by task type and then by increasing difficulty: exercises 1 and 3 to 6 in the recommended track (🎯); exercises 2 and 7 to 9 for going further (🏆). Write out each solution in full.

6.1. Brevet 1: Quick brevet-style questions 🎯

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- a) Expand: $(x + 6)(x - 6)$; $(2x - 5)(2x + 5)$.
- b) Factorise: $x^2 - 256$; $64x^2 - 9$.
- c) Give the opposite of $4x - 9$, then expand $-(6 - x) + 2(x - 1)$.

6.2. Brevet 2: Calculation program 🏆

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Consider the calculation program: choose a number; add 3; square the result; subtract the square of the starting number.

- a) Check that by choosing 2 at the start, we obtain 21.
- b) Express the result in terms of the starting number x , then expand and simplify.
- c) Which starting number gives the result 30?

6.3. Brevet 3: Expanding and proving a factorisation 🎯

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Let $E = (x - 3)^2 + (x - 3)(1 - 2x)$, where x denotes a number.

- a) Expand and simplify E .
- b) Prove that the factorised form of E is $(x - 3)(-x - 2)$.
- c) Calculate E for $x = 3$ by choosing the fastest form. Justify your choice.

6.4. Brevet 4: Clever calculation with justification 🎯

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Without a calculator, justifying each time with a notable identity:

- a) Justify that $104 \times 96 = 9984$.
- b) Calculate 1002^2 .
- c) Calculate $57^2 - 43^2$.

6.5. Brevet 5: Proof (parity) 🎯

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- a) Check that $7 + 9 = 16$ and $15 + 17 = 32$ are multiples of 4.
- b) Prove that the sum of two consecutive odd numbers is always a multiple of 4.
- c) Is the sum of any two odd numbers always a multiple of 4? Prove it, or refute it with a counterexample.

6.6. Brevet 6: The expression I 🎯

★★★☆☆

Let $I = (7x - 3)^2 - 25$.

- a) Expand and simplify I .
- b) Factorise I (difference of two squares: the first square is a bracketed expression).
- c) For $x = 1$, calculate I using the three available forms: the starting form, the expanded form and the factorised form. Compare the calculations and justify which form you would choose.
- d) For $x = \frac{3}{7}$, compare the starting form and the factorised form. Can one single form be declared always the best, whatever the value of x and whatever the purpose of the calculation?

6.7. Brevet 7: The garden path

★★★★

A square garden of side x (in metres) is surrounded by a path 3 m wide, forming a large outer square.

- Draw a diagram. Express the side of the large square in terms of x .
- Show that the area of the path is $12(x + 3)$.
- Calculate this area for $x = 17$ m: mentally, using the factorised form.
- Is the result of question c) plausible? Compare with the area of a 3 m by 80 m rectangle.

6.8. Brevet 8: The product of the two neighbours

★★★★

- Prove that for every integer n : $(n - 1)(n + 1) + 1 = n^2$.
- Deduce mentally: 59×61 ; 79×81 ; $999 \times 1\,001$.
- State the property proved in a) in a sentence beginning with “The product of the two integers immediately either side of the integer n , that is $n - 1$ and $n + 1$...”.

6.9. Brevet 9: Sum of the squares of two consecutive integers

★★★★

- Using a spreadsheet or by hand, calculate $n^2 + (n + 1)^2$ for $n = 0$, $n = 1$, $n = 2$, $n = 3$ and $n = 4$. Make a conjecture about the parity of the result.
- Prove this conjecture.
- Metacognitive question: in the proof, at what exact point can we see that the result is odd?

7. Interdisciplinary contextualised exercises

7.1. Physics: Braking energy

★★★☆☆

The kinetic energy (in joules, denoted J) of a vehicle of mass m (in kg) travelling at speed v (in m/s) is given by the formula $E_c = \frac{1}{2} m v^2$.

- Calculate E_c for a 1 200 kg car travelling at 10 m/s (i.e. 36 km/h).
- The speed doubles (20 m/s). Karim claims: “the energy doubles too”. Calculate it, then explain his error using $(2v)^2$.
- Generalise: if the speed is multiplied by k , what is the energy multiplied by? Which box from the course notes do we recognise here?

7.2. Urban planning: The reconfigured plot

★★★☆☆

A local council owns a square plot of land of side a (in metres, $a > 5$). A plan replaces it with a rectangular plot: 5 m longer and 5 m narrower, i.e. $(a + 5)$ by $(a - 5)$.

- The mayor announces: “the area is unchanged: what we gain on one side, we lose on the other”. Test with $a = 20$: what are the two areas?
- Express the area of the rectangle in terms of a and prove that the mayor is wrong: the plot always loses the same area. How much?
- Does this result depend on the size of the plot? Comment on it in one sentence (that sentence is the “answer”, not the calculation).

7.3. Programming: The identity tester

★★★☆☆

The Python script below has been written.

```
for x in range(-5, 6):  
    print(x, (x+5)**2 - x**2, 10*x + 25)
```

- What does this script calculate? Write the lines displayed for $x = -5$, $x = 0$ and $x = 2$.
- What conjecture does the full output suggest?
- This script tests 11 values. Is that a proof? Prove the conjecture by hand.
- Modify the script (on paper) to test the false claim “ $(x + 5)^2 = x^2 + 25$ ”: what is the only value of x that displays two equal numbers?

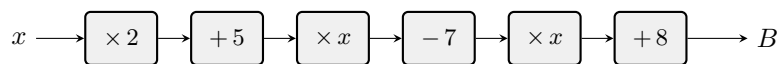
7.4. History of computation: Horner's machine

★★☆☆

To evaluate $B = 2x^3 + 5x^2 - 7x + 8$, a “naive” calculator computes each term separately, then adds them up. A form known as Horner's method makes it possible to nest the multiplications:

$$2x^3 + 5x^2 - 7x + 8 = ((2x + 5)x - 7)x + 8.$$

We can represent this calculation as a “machine” that chains the operations together:



- Run the machine for $x = 2$ (write down the result of each step), then calculate $2x^3 + 5x^2 - 7x + 8$ for $x = 2$ in the “naive” way. Compare.
- Prove that the machine never cheats: expand $((2x + 5)x - 7)x + 8$ and simplify.
- Count the multiplications: how many are needed in the naive way (reminder: $x^3 = x \times x \times x$)? And with the machine?
- Write $C = 3x^2 - 4x + 3$ in the same nested form, then evaluate C mentally for $x = 10$.

8. Going further

Bank of extension exercises, sorted by increasing difficulty. Exercise 8.1 is a pure reasoning challenge; 8.2 is a “proof without words” to decode; 8.3 anticipates the chapter on square roots (Prep. Year 11: the required property is provided). Exercises 8.4 to 8.6 extend the bank: a perfect square hidden in a product of four consecutive integers, factorisations of giant numbers (the arithmetic echo of 8.1) and a machine for generating Pythagorean triples.

8.1. Challenge: An almost prime square

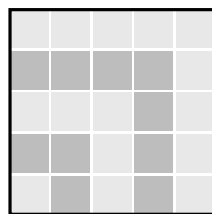
★★★★

Find all natural numbers n such that $n^2 - 9$ is a prime number. Hint: factorise. Can a product of two integers be prime?

8.2. Proof without words: The sum of odd numbers

★★★★

Look at the figure: each L-shaped “gnomon” adds a layer to the square.



$$1 + 3 + 5 + 7 + 9 = ?$$

- What is $1 + 3 + 5 + 7 + 9$? And $1 + 3 + 5 + \dots + 19$ (the first ten odd numbers)?
- What general property does the figure “prove without words”? State it using the letter n .
- For experts: check this property by calculation, assuming that the k -th odd number is $2k - 1$, and calculating $n^2 - (n - 1)^2$ (the area of the n -th L-shape).

8.3. Prep. Year 11: The conjugate expression

★★★★

This exercise uses the future number $\sqrt{7}$ (“square root of 7”), defined as the positive number whose square is 7: $\sqrt{7} \times \sqrt{7} = 7$ (and $\sqrt{7} \approx 2,65$). Everything else comes from this chapter.

We want to rewrite $B = \frac{1}{4 - \sqrt{7}}$ without a square root in the denominator.

- Calculate $(4 - \sqrt{7})(4 + \sqrt{7})$ using a notable identity and the given property.
- By multiplying the numerator and denominator of B by $(4 + \sqrt{7})$ (the conjugate expression of $4 - \sqrt{7}$), write B without a square root in the denominator.
- Which notable identity “makes it all work”? How does this method resemble the calculation of 102×98 ?

8.4. The hidden perfect square

★★★★

Calculate $1 \times 2 \times 3 \times 4 + 1$, then $2 \times 3 \times 4 \times 5 + 1$, then $3 \times 4 \times 5 \times 6 + 1$: each time, the result is a perfect square ($25 = 5^2$, $121 = 11^2$, $361 = 19^2$). Coincidence? Let n be a natural number; we set $a = n(n+3)$.

a) Expand a , then expand $(n+1)(n+2)$. Deduce that $(n+1)(n+2) = a+2$.

b) By cleverly grouping the four factors, show that

$$n(n+1)(n+2)(n+3) + 1 = a(a+2) + 1,$$

and then that this number is equal to $(a+1)^2$.

c) Conclude with a sentence (“the product of four consecutive integers, increased by 1, is always...”.) and state which number it is the square of, in terms of n . Check: recover 19^2 for $n=3$.

8.5. Giant prime factors (echo of 8.1)

★★★★

No long multiplication, no calculator: here, notable identities break down seven- or eight-digit numbers. (We assume that 1667, 4999 and 1009 are prime numbers.)

a) Notice that $A = 24\,999\,999 = 5\,000^2 - 1$. Deduce a factorisation of A , then its prime factorisation (one of the two factors obtained still hides a factor of 3).

b) Check that $B = 1\,018\,081$ can be written as $10^6 + 18 \times 10^3 + 81$. Recognise a perfect square (check the double product!), and deduce the prime factorisation of B .

c) As in 8.1: check that, for every integer n , $n^3 - 8 = (n-2)(n^2 + 2n + 4)$, then find all natural numbers n such that $n^3 - 8$ is a prime number.

8.6. The Pythagorean triple machine

★★★★

A Pythagorean triple is a trio of strictly positive integers (a, b, c) such that $a^2 + b^2 = c^2$: for example $(3, 4, 5)$, met in Year 9. Here is a machine for making as many of them as we like. Choose two integers u and v with $0 < u < v$, and set

$$a = v^2 - u^2, \quad b = 2uv, \quad c = v^2 + u^2.$$

a) Calculate the triple (a, b, c) obtained for $(u, v) = (1, 2)$, then $(2, 3)$, $(3, 4)$ and $(2, 5)$.

b) Check, on the first triple obtained, that $a^2 + b^2 = c^2$.

c) Prove that the machine always works: expand $(v^2 - u^2)^2 + (2uv)^2$ on the one hand, and $(v^2 + u^2)^2$ on the other hand, then conclude.

d) Make a new Pythagorean triple by choosing another pair (u, v) , and check it.

9. The bridge to chapter 02

Last page of the chapter. What follows asks a question that this chapter will not fully solve: that is deliberate. The general answer opens chapter 02.

9.1. Factorise... then solve?

★★☆☆

We are looking for all numbers x such that $x^2 - 9 = 0$.

a) Check that 3 works. Is it necessarily the only one?

b) Factorise $x^2 - 9$. The equation can then be written $(x+3)(x-3) = 0$: a product of two factors equals 0.

c) Find a second number that works, and check it.

d) The real question: a product of two numbers equals 0; what can be said about these two numbers? Is factorisation enough, using the properties already established in this chapter, to prove that there is no third solution? State precisely what has been checked, what is conjectured and what remains to be proved. The general property will be studied in chapter 02.

9.2. Ninon versus Grégory

★★★★

Ninon and Grégory want to solve the equation $(x-1)^2 = 9$.

• Ninon: “The numbers whose square is 9 are 3 and -3 . So I solve $x-1=3$ and $x-1=-3$.”

• Grégory: “The equation can also be written $(x-1)^2 - 9 = 0$. I factorise the left-hand side and get a product equal to 0, as in 9.1.”

a) Carry Ninon’s method through to the end, then factorise like Grégory. Which values do the two forms bring out?

b) Check these values in the starting equation. Is factorisation already enough to prove that there are no others?

c) Which of the two methods reuses the factorisation from this chapter? Which property is still missing to use the zero product rigorously? (Full answer in chapter 02.)