



Exercise sheet

Similar triangles: Enlargements and reductions

Difficulty: ★☆☆☆ very easy ★★☆☆ easy ★★★☆ medium ★★★★ hard

Track: 🏃 core 🎯 recommended 🏰 going further

The “Optional enrichment” blocks placed after 1.2, 2.5, 4.4, 6.8, 7.7 and 7.8 are to be tackled after the initial exercise, to go further.

Common framework. Two triangles are similar when the lengths of their sides are proportional; the constant k is the scale factor. The sides that match each other are corresponding sides.

Angle characterisation. Two triangles are similar if, and only if, their angles are equal in pairs, and two pairs of equal angles are enough (the third is 180° minus the other two).

Where to find equal angles? Marked in the problem • common angle • vertically opposite angles • alternate interior or corresponding angles formed with parallel lines • calculating the third angle.

Enlargement–reduction with scale factor k : lengths and perimeters are multiplied by k , angles keep their size, areas are multiplied by k^2 , volumes by k^3 . A scale of $\frac{1}{n}$ is a reduction with scale factor $k = \frac{1}{n}$.

Reverse direction: if the problem gives the areas, their ratio is k^2 ; we find k by looking for the positive number whose square is equal to this ratio.

Classic pitfalls:

- Do not confuse “similar” and “congruent”: two similar triangles have the same shape, not necessarily the same size (two congruent triangles are similar with $k = 1$).
- The scale factor is read between corresponding sides (matched up using the equal angles), never between sides picked at random.
- Areas are multiplied by k^2 , never by k ; volumes by k^3 .
- A single pair of equal angles is not enough; two pairs of equal angles are needed.
- Watch the direction of k : multiply by k in the direction used to calculate it, divide by k in the other; check that the result makes sense (the smaller triangle must have the shorter sides).

Basic method, the three key steps:

1. Match up: identify the corresponding vertices using the equal angles (opposite equal angles lie the corresponding sides).
2. Set up: proportionality table of the corresponding sides, one triangle per row.
3. Solve: calculate k from a complete column, then multiply or divide by k (or cross-multiply). Check: are all three ratios equal? is the longest side opposite the largest angle?

Fluency (no calculator).

- | | |
|---|---|
| a) Two angles of a triangle: 47° and 86° . The third? | g) Simplify $\frac{3}{4,5}$. |
| b) Two angles of a triangle: 90° and 26° . The third? | h) Reduction with scale factor 0,5: a length of 7 cm becomes... |
| c) Angles 40° and 60° here, 60° and 80° there: similar? | i) $k = 1,2$ and perimeter 15 cm: new perimeter? |
| d) Enlargement with scale factor 3: the area is multiplied by... | j) Scale $\frac{1}{100}$: 3 cm on the plan represent... |
| e) Enlargement with scale factor 2: the volume is multiplied by... | |
| f) The area has been multiplied by 25: what is k ? | |

Retrieval practice (previous chapters: 5 questions).

1. (Ch. 00) Write 360 as a product of prime factors.
2. (Ch. 01) Write $\frac{54}{36}$ as a fraction in lowest terms.
3. (Ch. 01) Calculate $\frac{3}{5} \times \frac{10}{9}$.
4. (Ch. 01) Calculate $(0,5)^3$ and $\left(\frac{2}{3}\right)^2$.
5. (Ch. 01) Write $1,25 \times 10^4$ as an ordinary number.

1. Recognising similar triangles

1.1. Similar or not? (using angles) 🐦

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In this set of questions, we compare a triangle ABC and a triangle DEF .

In each case, state whether these two triangles are similar. Justify your answer by calculating the third angle when necessary.

- a) In ABC , $\hat{A} = 47^\circ$ and $\hat{B} = 86^\circ$. In DEF , $\hat{D} = 47^\circ$ and $\hat{E} = 47^\circ$.
- b) In ABC , $\hat{A} = 35^\circ$ and $\hat{B} = 72^\circ$. In DEF , $\hat{D} = 73^\circ$ and $\hat{E} = 35^\circ$.
- c) In ABC , $\hat{A} = 50^\circ$ and $\hat{B} = 60^\circ$. In DEF , $\hat{D} = 60^\circ$ and $\hat{E} = 65^\circ$.
- d) ABC is right-angled at A with $\hat{B} = 32^\circ$. DEF is right-angled at D with $\hat{E} = 58^\circ$.
- e) In ABC , $\hat{A} = 44^\circ$ and $\hat{B} = 68^\circ$. DEF is isosceles with apex D and $\hat{D} = 92^\circ$.
- f) ABC is equilateral. In DEF , $\hat{D} = 60^\circ$ and $\hat{E} = 60^\circ$.

1.2. Similar or not? (using sides) 🐦

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- a) In each case, are triangles ABC and DEF similar? Justify using the three ratios.
 - ABC ($AB = 3$; $BC = 4$; $AC = 5$) and DEF ($DE = 6$; $EF = 8$; $DF = 10$).
 - ABC ($AB = 3$; $BC = 4$; $AC = 5$) and DEF ($DE = 4$; $EF = 5$; $DF = 6$).
 - ABC ($AB = 4,5$; $BC = 6$; $AC = 7,5$) and DEF ($DE = 3$; $EF = 4$; $DF = 5$).
- b) Think critically. ABC and EFG are two triangles such that: $AB = 5$ cm, $AC = 8$ cm and $BC = 6,5$ cm; $EF = 1$ cm, $EG = 1,6$ cm and $FG = 1,2$ cm.
 - Is triangle EFG a reduction of triangle ABC ?
 - If not, change one length so that it is. State the scale factor of the reduction.

1.2. Optional enrichment 🧑🏫

Does adding the same length preserve the shape?

In the initial exercise, the triangles with sides $(3; 4; 5)$ and $(4; 5; 6)$ are not similar. Yet 1 was added to each length. Does this always fail? We want to find out exactly which triangles keep their shape when the same positive length is added to all three sides. All measurements are in centimetres.

Useful concepts before starting

Adding t to a length a gives $a + t$. Multiplying this length by k gives ka . To go from 4 to 6, we add 2, whereas we multiply by $6/4 = 1,5$. The scale factor is a quotient; the increase is a difference.

Given the sides alone, we order both lists from smallest to largest before comparing their quotients. Adding the same number preserves this order. Three lengths $0 < a \leq b \leq c$ form a non-degenerate triangle if and only if $a + b > c$.

We will use the distributive property $u(v + w) = uv + uw$, for example $3(4 + 2) = 12 + 6$. In an equation, we can subtract the same number from both sides and divide both sides by the same non-zero number. An exact condition requires showing first that every solution must satisfy it, and then that it is indeed sufficient.

A. Formulating a conjecture

A1) Add 1 to each side, then fill in the three quotients “new side / old side”.

Initial sides	New sides	Quotient 1	Quotient 2	Quotient 3
(3; 4; 5)				
(5; 5; 8)				
(4; 4; 4)				

A2) Check that the three new lists do indeed define triangles. Which of them are similar to their starting triangle? Justify using the exact values of the quotients.

A3) What special feature of the initial triangle seems to allow the shape to be preserved? State a conjecture. Why are the three trials not enough to prove it?

B. Proving a necessary and sufficient condition

We start with a triangle with sides $0 < a \leq b \leq c$ and fix a number $t > 0$. The new sides are $a + t, b + t, c + t$.

- B1) Show that these lengths are positive, arranged in the same order, and always form a non-degenerate triangle. Use $a + b > c$ to compare $(a + t) + (b + t)$ and $c + t$.
- B2) Suppose that the triangles are similar. Then $(a + t)/a = (b + t)/b$. Cross-multiply, expand, subtract ab , then divide by t . What relation do we obtain between a and b ? Repeat with a and c . What type of triangle is therefore necessary?
- B3) Conversely, if the triangle is of this type, check that the three quotients are equal for all $t > 0$. Give the scale factor from the starting triangle to the new triangle and conclude with an “if and only if” sentence.

C. Changing the hypothesis

- C1) If $t = 0$, which triangles work? Which step in B2 is then forbidden? Why was it necessary to specify $t > 0$?
- C2) We now subtract 2 from each side of the triangle (3; 4; 5). All three results are positive: is checking this enough to obtain a triangle? Compare this situation with B1 and explain why an addition rule does not automatically carry over to subtraction.

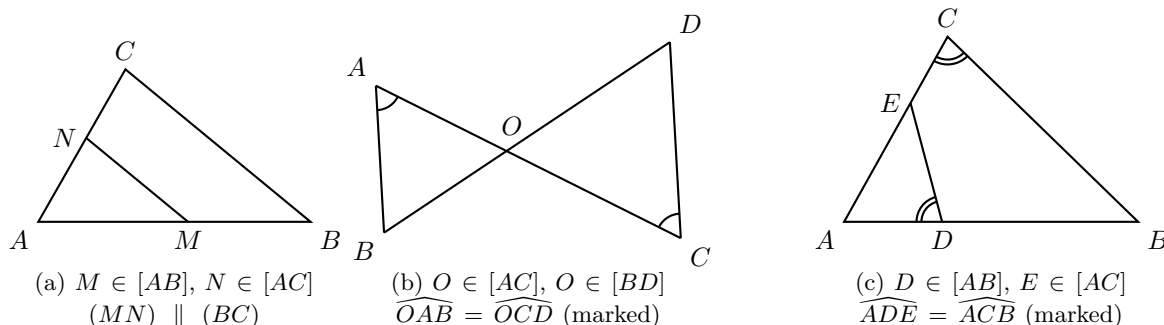
To remember

An enlargement requires a single common multiplier. Adding the same amount does not have the same relative effect on a short side and on a long side. For any proposed transformation, both proportionality and the existence of the triangles must be checked.

1.3. Justifying similarity: three configurations

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For each figure, prove that the two stated triangles are similar, using the standard wording from the course notes (two pairs of equal angles, each justified).

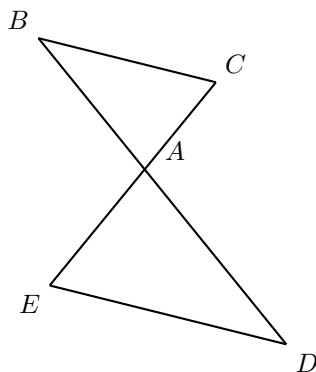


- a) Figure (a): prove that triangles AMN and ABC are similar.
- b) Figure (b): prove that triangles OAB and OCD are similar.
- c) Figure (c): prove that triangles ADE and ACB are similar, then state the three pairs of corresponding vertices. Careful: here, does E correspond to $C \dots$ or to B ? Be guided by the angles, not by the position.

1.4. The bowtie configuration

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The segments $[BD]$ and $[CE]$ intersect internally at A , and the lines (BC) and (DE) are parallel. We are given $AB = 3,6$ cm and $AD = 4,8$ cm.



- a) What can be said about triangle ADE in relation to triangle ABC ? Justify using the angles (no intercept theorem (Thales' theorem in France) here!), and state the scale factor from ABC to ADE .
- b) Find AE given that $AC = 2,4$ cm.

c) Find BC given that $DE = 5,2$ cm.

1.5. True or false? ★★★★☆

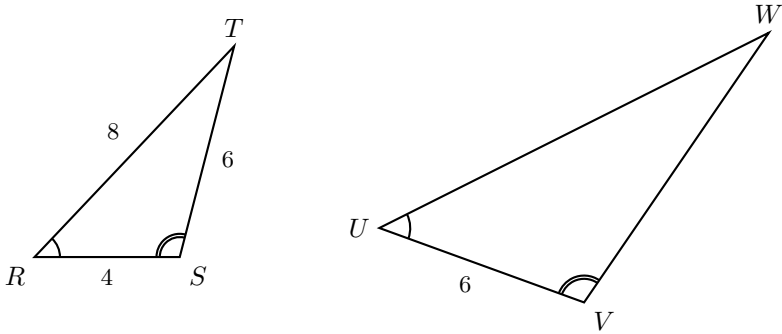
For each statement: true or false? A “true” statement must be proved (using the angles); a “false” statement must be disproved with a numerical counterexample.

- a) Two equilateral triangles are always similar.
- b) Two isosceles triangles are always similar.
- c) Two right-angled triangles are always similar.
- d) Two right-angled isosceles triangles are always similar.
- e) Two isosceles triangles each having an 80° angle are always similar.
- f) Two similar triangles are always congruent.
- g) Two congruent triangles are always similar.

2. Calculating lengths

2.1. First calculation, two ways ★★☆☆

Triangles RST and UVW are similar, with the correspondence $R \leftrightarrow U, S \leftrightarrow V, T \leftrightarrow W$ (angles marked on the figure). We are given: $RS = 4$; $ST = 6$; $RT = 8$; $UV = 6$.

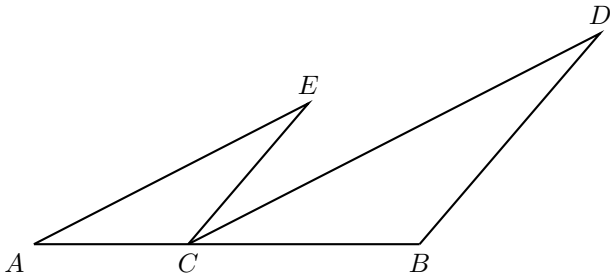


- a) Give the scale factor k from RST to UVW .
- b) Calculate VW in two ways: using the scale factor k , then by cross-multiplying. Which of the two is quicker here, and why?
- c) Calculate UW .
- d) Calculate the perimeter of each triangle. What do you notice? Was this predictable?

2.2. The brevet exercise (French national exam at the end of Year 10) ★★★★☆

This is the classic exercise of the chapter: you should be able to do it with your eyes closed.

In the figure below: point C lies on segment $[AB]$; $AC = 3$; $AB = 7,5$; $BD = 5,4$ and $CD = 9$; lines (AE) and (CD) are parallel; lines (CE) and (BD) are parallel.

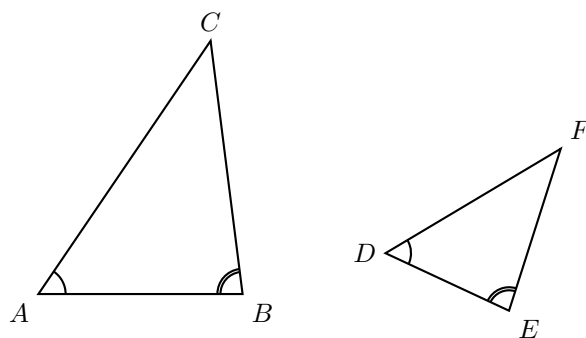


- a) Prove that angles $\widehat{BCD}$ and $\widehat{CAE}$ are equal.
- b) Prove that triangles ACE and CBD are similar.
- c) Hence find the lengths of the sides of triangle ACE .

2.3. Corresponding sides: two configurations

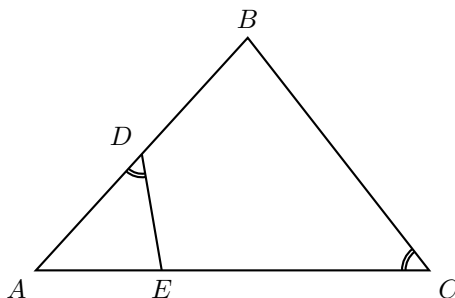
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Configuration 1. Triangles ABC and DEF below have their angles marked: $\widehat{A} = \widehat{D}$ and $\widehat{B} = \widehat{E}$. We are given: $AB = 6$; $BC = 7,5$; $AC = 9$; $DE = 4$.



- Justify that triangles ABC and DEF are similar.
- Draw and complete a table of corresponding sides, then give the scale factor from ABC to DEF .
- Calculate EF and DF .

Configuration 2. In triangle ABC , D is a point on $[AB]$ and E is a point on $[AC]$ such that $\widehat{ADE} = \widehat{ACB}$ (marked on the figure). We are given: $AD = 4$; $AC = 10$; $AB = 8$; $CB = 7,5$.



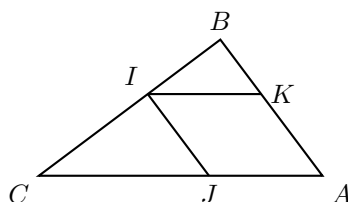
- Prove that triangles ADE and ACB are similar.
- State the corresponding sides. Pitfall ahead: $[AD]$ is not compared with $[AB]$...
- Calculate DE and AE .

2.4. The 3-4-5 triangle

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(Unit of your choice.) Let ABC be a triangle such that $AB = 3$, $BC = 4$ and $CA = 5$.

- Prove that ABC is right-angled at B .
- Let $J \in [CA]$ be such that $CJ = 3$. The line parallel to (AB) passing through J intersects (BC) at I ; the line parallel to (CA) passing through I intersects (AB) at K . Copy the figure below at full size and complete it.



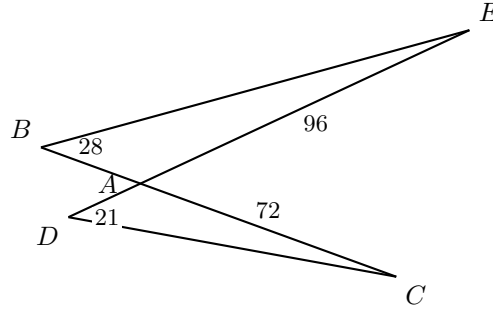
- Prove that triangles JIC and ABC are similar (using the angles), give the scale factor from ABC to JIC , then calculate CI and JI .
- Hence find BI , then prove that triangles KBI and ABC are similar and calculate KB and KI .
- Calculate the area of triangle ABC , then, without calculating any other lengths, the areas of triangles KBI and JIC (think of k^2). Hence find the area of quadrilateral $AKIJ$.

2.5. The butterfly with proportional sides

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SAS follow-up to the course notes, section 2.3: redo the example without the course notes, then extend it.

The line segments $[BC]$ and $[DE]$ intersect in their interior at A . We are given: $AB = 28$ mm; $AC = 72$ mm; $AD = 21$ mm; $AE = 96$ mm. Care: no pair of parallel lines is given.



- Prove that triangles DAC and BAE are similar. Which criterion from the course notes applies here, and why was the angle criterion not sufficient on its own?
- What is the ratio of the areas, from DAC to BAE ? State which of the two has the larger area.
- In exercise 1.4, the “wings” of the butterfly were parallel. Are the lines (DC) and (BE) parallel here? Justify. (Hint: if they were, which alternate interior angles would be equal for the transversal (BC) ? Compare with what the similarity in a) tells us.)

2.5. Optional enrichment

Why must the angle be included between the sides?

The SAS criterion in the course notes uses two proportional sides and the included angle between those sides. Can we drop the word “included”? We will construct and check a counterexample, then repair the criterion.

Useful concepts before starting

The included angle between sides $[AB]$ and $[BC]$ is the angle with vertex B , denoted $\widehat{ABC}$: the two sides meet there. The angle $\widehat{BAC}$ has vertex A ; it is not included between $[AB]$ and $[BC]$.

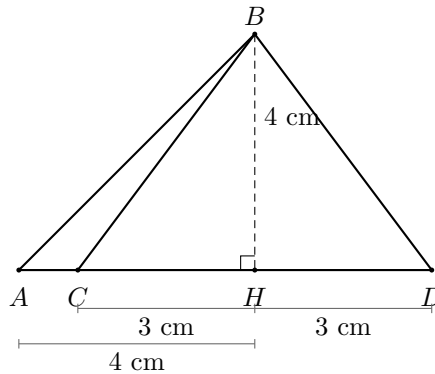
A counterexample satisfies all the hypotheses of a statement, but not its conclusion. A single one is enough to show that the statement is not always true. For example, 1, 2, 4 are three positive lengths, but they do not form a triangle, because $1 + 2 < 4$.

To test sides on their own, we order them from shortest to longest in each triangle: if they are proportional, the three corresponding quotients are equal. Example: $(3; 4; 5)$ and $(6; 8; 10)$ give $6/3 = 8/4 = 10/5 = 2$.

We will use Pythagoras’ theorem and the square root: $\sqrt{u}$ is the positive number whose square is $u > 0$. Thus $\sqrt{25} = 5$; since $25 < 30 < 36$, we have $5 < \sqrt{30} < 6$.

A. Construct the exact data of the counterexample

The points A, C, H, D are collinear in that order. We are given $AH = BH = 4$ cm, $HC = HD = 3$ cm and $(BH) \perp (AD)$. The two triangles studied are ABC and ABD .



- Calculate AC and AD from the order of the points.
- Apply Pythagoras’ theorem in BHC , BHD , then BHA . Deduce BC , BD and AB . Justify the bounds $5 < AB < 6$ without using a rounded value.
- What type of triangle is ABH ? Deduce the size of $\widehat{BAH}$, then justify that $\widehat{BAC} = \widehat{BAD} = 45^\circ$.

B. Test similarity without relying on the drawing

B1) Complete the following table with the names of the sides and their exact lengths:

Triangle	Shortest side	Middle side	Longest side
ABC			
ABD			

- B2) If the triangles were similar, what scale factor from ABC to ABD would the shortest sides give? Could the quotient obtained with the middle sides have this value? Conclude and justify.
- B3) Now check that $AB/AB = BC/BD = 1$ and that the angles at A are equal. Why does the SAS criterion nevertheless fail to apply? Write out the false statement that this figure disproves.

C. Repair the hypotheses and transfer

- C1) We want to compare sides $[AB]$ and $[BC]$ with sides $[AB]$ and $[BD]$. Which angle equality would we have needed to know in order to use the SAS criterion?
- C2) Another triangle EFG satisfies $EF = 2AB$, $FG = 2BC$ and $\widehat{EFG} = \widehat{ABC}$. Justify that it is similar to ABC , specify the corresponding vertices and calculate EG . Explain what distinguishes this situation from the counterexample.

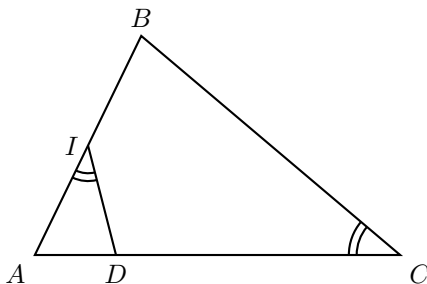
To remember

Equal ratios are not enough on their own. In the SAS criterion, the known angle must be the one formed by the two compared sides: the word “included” is an essential hypothesis.

2.6. The common angle, the midpoint and the areas 🏰

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In triangle ABC , we are given $AB = 28$ mm, $BC = 39$ mm and $AC = 42$ mm. I is the midpoint of $[AB]$, and D is the point on $[AC]$ such that $\widehat{AID} = \widehat{ACB}$ (angles marked on the figure).



- Prove that triangles AID and ACB are similar, and specify the three pairs of corresponding vertices.
- Calculate AD and ID (exact values; write AD as a fraction).
- Prove that $\frac{\text{area}(AID)}{\text{area}(ABC)} = \frac{1}{9}$.

3. Enlargements and reductions

3.1. Drills: the effects of k , k^2 , k^3 🦋

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No calculator: almost everything can be done mentally.

- $AB = 12$ cm, $AC = 8$ cm, $BC = 6$ cm, and triangle AMN is a reduction of triangle ABC with scale factor 0,25. Mentally calculate the perimeter of triangle AMN .
- The area of a triangle is 25 cm^2 . It is enlarged with scale factor 2: what is the new area?
- A cube has a volume of $27\,000 \text{ m}^3$. A scale model is made at a scale of $\frac{1}{30}$: what is the volume of the model?
- The area of a shape has been multiplied by 9: what was the scale factor of the enlargement?
- The lengths of a solid are multiplied by 5: the volume is multiplied by...
- Multiple choice. A camera screen is a rectangle of length 5,4 cm; the printed photo has length 15 cm. The scale factor of the enlargement is: A. 2,7 B. 2,8 C. $\frac{25}{9}$

g) Multiple choice. A reduction could for instance have scale factor: A. 120% B. $\frac{4}{3}$ C. $\frac{3}{4}$

h) In an enlargement with scale factor k , angle sizes are multiplied by... (careful!)

Application 3.1 bis: the photographed pennant

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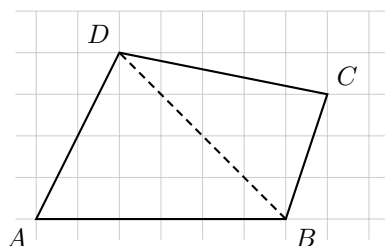
In a photograph, a triangular pennant has one side of 4 cm and another of 3 cm. The real pennant, similar to the one in the photograph, has a length of 60 cm for the side corresponding to the 4 cm side.

- What is the scale factor of the enlargement from the photograph to the real pennant?
- What is the real length of the side corresponding to the 3 cm side? Check that the result makes sense.

3.2. The quadrilateral: conjecturing with k and k^2

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$ABCD$ is the quadrilateral below, drawn on a 1 cm square grid; the diagonal $[BD]$ is drawn.



- On a grid, construct a reduction $A'B'C'D'$ of this quadrilateral with scale factor 0,5.
- How does the size of angle $\widehat{D'A'B'}$ compare with that of $\widehat{DAB}$? (No protractor needed.)
- Calculate the area of triangle ABD , then that of triangle $A'B'D'$. Copy and complete: $\mathcal{A}' = 0,5 \cdots \times \mathcal{A}$.
- Without drawing it, give the area of triangle $A''B''D''$ for an enlargement $A''B''C''D''$ with scale factor 1,2.
- Copy and complete the conjecture: “In an enlargement or a reduction with scale factor k , lengths are..., angles are..., areas are...”.

3.3. Working backwards: finding k

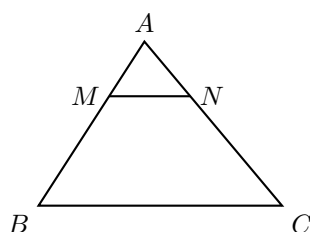
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- Argue carefully. A' and B' are points on the rays $[CA)$ and $[CB)$ such that (AB) and $(A'B')$ are parallel, with $AB = 1,5$ cm and $A'B' = 2$ cm. The area of triangle ABC is $2,7$ cm². Prove that triangles CAB and $CA'B'$ are similar (using angles), then calculate the area of triangle $CA'B'$.
- The area of a shape is 72 cm². After enlargement, the resulting shape has an area of 1800 cm². What is the scale factor of this enlargement?
- Multiple choice. A rectangle $A'B'C'D'$ of area 24 cm² is the enlargement with scale factor 1,25 of a rectangle $ABCD$. The area (in cm²) of $ABCD$ is: A. 15,36 B. 19,2 C. 30 D. 37,5
- On a map at a scale of $\frac{1}{3000}$, a football pitch has an area of 8 cm². What is the area (in m²) of this pitch in reality? Is the result plausible for a football pitch?

3.4. Is Arthur right?

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We know that: the points A, M, B are collinear in that order, as are A, N, C ; the lines (MN) and (BC) are parallel; $AB = 6$ cm and $AM = 2$ cm; the area of triangle ABC is 9 cm².



- Arthur claims: “Triangle AMN is a reduction of triangle ABC ”. Is he right? Explain, and if so give the scale factor of the reduction.
- Deduce the area of triangle AMN .

3.5. The Statue of Liberty: a critical look

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The Statue of Liberty in New York, with a height (excluding the pedestal) of 46 m, was designed by the French sculptor Auguste Bartholdi (1834–1904). A replica stands on the Île aux Cygnes in Paris; its height is 11,50 m.

- What is the scale factor of the reduction from the New York statue to the Paris statue?
- Take a critical look. The mass of a statue is linked to the volume of the materials used. To test a simplified model inspired by these statues, we use masses of 225 tonnes and 14 tonnes here; these are the conventional data for this exercise. Are these masses compatible with a reduction of scale factor $1/4$ of all dimensions, including thicknesses, using the same materials in the same places? Justify by a calculation. Can we deduce that the outer shapes are not similar?

3.6. Ice cream cones

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A company offers three sizes of ice cream cones of the same shape: large, with a volume of 720 mL, medium and small.

- The medium cone is a reduction of the large one with scale factor 75 %. Calculate its volume.
- The small cone is a reduction of the large cone with scale factor 50 %. Marie claims: “The volume of the small cone is equal to half that of the medium cone.” Is she right? Justify with a full calculation.
- Synthesis question: by what percentage does the volume decrease when the lengths decrease by 25 %? State the takeaway of the exercise in one sentence.

3.7. Doubling a cube: the Delian problem

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(The positive number whose cube is 2 is written $\sqrt[3]{2}$, which reads: “cube root of 2”; we are given $\sqrt[3]{2} \approx 1,26$.)

A cube has an edge of 10 cm. We want to make a second cube whose volume is double that of the first (the ancient problem of the altar of Delos).

- Calculate the volume of the starting cube.
- Léa suggests: “you just need to double the edge, so take 20 cm”. Calculate the volume of a cube with edge 20 cm. Is Léa right? By what factor has she actually multiplied the volume?
- By what number k must the edge be multiplied for the volume to be multiplied by 2? Deduce the edge of the second cube, to the nearest millimetre (use $\sqrt[3]{2} \approx 1,26$).
- Check: calculate the volume of a cube with this edge, and observe that it is indeed (approximately) double the starting volume.

3.8. Scratch: two triangles of the same shape

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Add the Pen extension and create the variable **side**. The unit of length is the Scratch step.

- Predict. Fill in the two missing numbers in the script to draw an equilateral triangle of side 60 steps.

```
When green flag clicked:
  pen up; erase all; go to  $x = -180$ ,  $y = 60$ ; point in direction  $90^\circ$ ;
  set side to 60; pen down;
  repeat  times:
    move side steps; turn right  degrees;
  pen up.
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- Program and run. Build the completed script, then run it. Check that the triangle closes. Explain why the turn angle is not 60° .
- Enlarge. At the end of the script, without erasing the first drawing: go to $x = 20$, $y = 60$, point in direction 90° , set **side** to 90, pen down, reuse the same loop, then pen up. Predict the scale factor of the enlargement, then run the script.
- Justify. Why are the two triangles similar? Calculate both perimeters. By what factor is the area multiplied from the first to the second triangle? Keep the program and a screenshot of the two triangles, together with these justifications.
- Apply again. Replace 90 with 30. Predict the scale factors on the lengths, the angles, the perimeter and the area of the second triangle compared with the first; check by running the script.

4. Problems

4.1. The triangular park to scale

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A triangular playground DEF is drawn to scale. Real dimensions: $DE = 12$ m, $EF = 9$ m, $DF = 15$ m. On the drawing, $DE = 6$ cm.

- Before any calculation: make a freehand sketch showing the real measurements and the one on the drawing.
- What is the scale of the drawing?
- Give the lengths EF and DF on the drawing.
- Prove that the real park is right-angled at E , then calculate its real area.
- Is the area of the drawn triangle equal to the real area divided by 200? Justify without calculating the area of the drawing, then check by calculation.

4.2. The 1/125 model aeroplane

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A scale model of a very large airliner is made at a scale of $\frac{1}{125}$.

- The length of the aeroplane is 73 m. What is the length of the model? Give the answer in centimetres.
- The wing area of the model is $540,8 \text{ cm}^2$. What is the wing area (in m^2) of the aeroplane?
- The fuel tank of the aeroplane holds 310 000 L. What is the capacity (in cm^3) of the model's tank?
- Is each result plausible? (One sentence per question.)

4.3. The cascade of parallels

★★★★

Let ACB' be a triangle, and let $B \in [AC]$ be a point distinct from A and C , closer to A than to C .

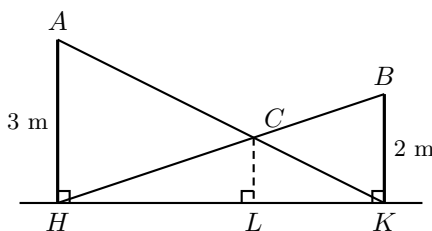
- The line parallel to (BB') through C meets (AB') at C' .
 - The line parallel to $(B'C)$ through C' meets (AC) at D .
- Draw a figure (triangle “not too large”, B closer to A than to C) and complete it.
 - Prove that triangles ABB' and ACC' are similar, and then that $\frac{AB}{AC} = \frac{AB'}{AC'}$.
 - Similarly prove, using triangles $AB'C$ and $AC'D$, that $\frac{AB'}{AC'} = \frac{AC}{AD}$.
 - Deduce that $\frac{AB}{AC} = \frac{AC}{AD}$.
 - Suppose $AB = 1$ and write $AC = x$. Show that $AD = x^2$. How can the construction be continued to obtain a segment of length x^3 ? (We say that AC is the geometric mean of AB and AD : a machine for multiplying by x , invented long before the calculator.)

4.4. The two poles

★★★☆☆

Two vertical poles $[HA]$ (height $AH = 3$ m) and $[KB]$ (height $BK = 2$ m) stand on horizontal ground, their bases H and K being a few metres apart. A cable is stretched from the top A to the base K , and a cable from the top B to the base H : the cables cross at C . Let L be the point on the ground vertically below C : the points H, L, K are collinear and $(CL) \perp (HK)$.

At what height CL do the two cables cross?



- Justify that the lines (AH) , (CL) and (BK) are parallel.
- Prove that triangles KLC and KHA are similar, and deduce that $\frac{CL}{3} = \frac{KL}{KH}$.
- Similarly prove that $\frac{CL}{2} = \frac{HL}{HK}$.

- d) By adding the two previous equalities, calculate CL .
- e) Re-read the problem: a piece of information that we might have expected to need was never used. Which one? What does this mean in practice for the two poles?

4.4. Optional enrichment 🏠

Choosing the poles: finding all solutions

In the original exercise, the heights of the poles were given. Now we reverse the problem: the cables must cross 2 m above the ground, and the heights of the poles must be whole numbers of metres. Which choices are possible?

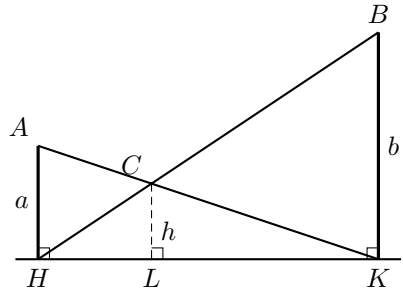
Useful concepts before starting

We use the same configuration as in exercise 4.4. The poles are vertical on horizontal ground; their bases are distinct. The cables join each top to the opposite base. The positive numbers a, b, h denote the lengths, in metres, of AH, BK, CL .

The reciprocal of a positive number u is $1/u$; their product is 1. Example: if $1/b = 2/3$, then $b = 3/2$, because $(2/3)(3/2) = 1$. This result is a possible height, but it is not a whole number of metres.

For positive numbers, taking the reciprocal reverses the order: the larger the denominator, the smaller the fraction with numerator 1. For example, $5 > 2$ and $1/5 < 1/2$.

We do not distinguish a choice of poles from the one obtained by swapping left and right. We will therefore call a the smaller of the two heights, allowing equality: $a \leq b$.



A. Obtain a relation valid for all heights

- A1) Use the two pairs of similar triangles from the original exercise again. Establish $h/a = KL/KH$ and $h/b = HL/HK$, then add them to obtain $h/a + h/b = 1$. Why can we then divide by h ? Deduce

$$\frac{1}{h} = \frac{1}{a} + \frac{1}{b}.$$

- A2) Check that with poles of 3 m and 6 m, the crossing point is 2 m above the ground. Does the distance between the bases change this result?
- A3) Since $1/b > 0$, compare $1/h$ and $1/a$, then h and a . Do the same with b . What does this result mean on the figure?

B. Narrow down the search so nothing is missed

We now set $h = 2$, with a and b positive integers and $a \leq b$.

- B1) Using A3, explain why a can be neither 1 nor 2.
- B2) Suppose $a \geq 5$. Why would we also have $b \geq 5$? Show that then $1/a + 1/b \leq 2/5 < 1/2$, which is impossible. What are the only integer values of a left to examine?
- B3) For each one, complete the table. First calculate $1/b$, then take its reciprocal to find b .

a	$1/b = 1/2 - 1/a$	b	Integer and $b \geq a$?
3			
4			

- B4) Check each pair obtained in the relation from A1. Give all choices of heights, without counting a swapped pair twice. Explain separately why each one works and why no other is possible.

C. Transfer: require a crossing at 3 m

- C1) We keep integer heights and $a \leq b$, but we set $h = 3$. Find all possible pairs. Start by bounding a , then draw up a table giving $1/b$ and b . A fractional candidate does not satisfy the initial constraint.
- C2) Among these choices, we want the taller pole to be as short as possible. Which choice should we keep? Is it the one that uses the shortest possible pole? Justify both answers.

To remember

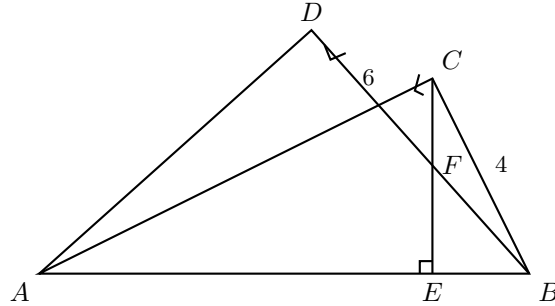
A formula makes it possible to test choices. To find all solutions, we must also narrow down the search, examine all remaining cases and check the constraints. Looking for an example, looking for all solutions and choosing the best one according to a criterion are three different tasks.

4.5. Cascade of similarities: altitudes in action

★★★★

ABC and ABD are two triangles right-angled at C and D respectively, drawn on the same side of their common hypotenuse $[AB]$, with $BC = 4$ and $BD = 6$. In triangle ABC , E is the foot of the altitude from C . The segment $[CE]$ intersects the segment $[BD]$ at F .

Goal: calculate BF , without ever knowing AB .



- Prove that triangles ABC and CBE are similar, then deduce that $BC^2 = BE \times BA$.
- Prove that triangles ABD and FBE are similar, then deduce that $BF \times BD = BE \times BA$.
- Deduce from questions a) and b) that $BF = \frac{BC^2}{BD}$, then calculate BF (exact value).
- Question for understanding: the length AB was not given, and yet BF is completely determined. What happens to BF if we redraw the figure with a larger AB (still with $BC = 4$ and $BD = 6$)?

5. Justifying and reasoning

5.1. Spotting the error: Naïm's work

★★☆☆

Triangles ABC and DEF are similar, with $\hat{A} = \hat{D}$ and $\hat{B} = \hat{E}$. We are given $AB = 4$, $BC = 6$ and $DE = 6$. We need to calculate EF .

Naïm's work. " $BC = 6$ and $DE = 6$, so the scale factor is $\frac{6}{6} = 1$: the triangles are congruent, and $EF = BC = 6$."

- Naïm's answer is wrong. Identify the exact mistake made (which step of the method was skipped?).
- Write out the correct solution and give the true value of EF .

5.2. Checking consistency: without calculating

★★☆☆

Each of the following statements is false, and this can be seen without writing down a single calculation. Explain why, in one sentence each time.

- "An enlargement with scale factor 3 multiplied the area of the shape by 6."
- "In my triangle reduced at a scale of $\frac{1}{2}$, the 8 cm side became a 16 cm side."
- "These two triangles are similar: in the first, the longest side is opposite the largest angle; in the second, which is not equilateral, the longest side is opposite the smallest angle."
- "The scale model is at a scale of $\frac{1}{100}$, so the surface area of paint needed is 100 times smaller than for the real object."

5.3. Writing a complete solution

★★★★

The segments $[AC]$ and $[BD]$ intersect at O . We know that $\widehat{OAB} = \widehat{OCD}$, and we are given: $OA = 6$; $OC = 4$; $OB = 5,4$; $CD = 5$.

Write a complete, self-contained solution following the template practised in the chapter:

- prove that triangles OAB and OCD are similar (two pairs of equal angles, each justified);
- match the vertices, set up the table, then calculate OD and AB ;
- check (equal ratios, large/small consistency) and conclude with a sentence.

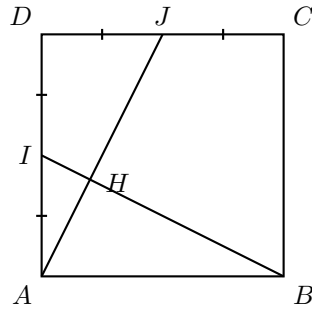
1. Modify and predict. In the previous problem, replace $OC = 4$ with $OC = 3$, without changing OA , OB or CD . Before calculating, predict the effect on the scale factor from OAB to OCD , on OD and on AB , then check.
2. Create under constraints. Invent a problem about two similar triangles, the second being a reduction of the first with scale factor $\frac{3}{4}$, with one unknown length in each triangle and enough data to calculate them. State the correspondence of the vertices unambiguously.
3. Solve, check, swap. Solve the problem, justifying the similarity and the matching of vertices, check all three ratios, then have a classmate solve your problem. Revise any unnecessary, missing or ambiguous data.

5.4. Proving perpendicularity: the two midpoints of the square

★★★☆☆

$ABCD$ is a square. I is the midpoint of $[AD]$ and J is the midpoint of $[DC]$. The segments $[AJ]$ and $[BI]$ intersect at H .

Goal: prove that $(AJ) \perp (BI)$, without a protractor, without measuring, without a grid.



- Prove that the triangles ABI and DAJ are congruent (state precisely the congruence criterion used: section 1 of the course notes).
- Deduce that $\widehat{ABI} = \widehat{DAJ}$ and that $\widehat{AIB} = \widehat{DJA}$.
- Prove that the triangles AHB and JDA are similar.
- Conclude that $\widehat{AHB} = 90^\circ$.
- Alternative method. Obtain the result again by calculating the sum $\widehat{HAB} + \widehat{ABH}$ in triangle ABH . Compare the two methods: which one is shorter, and why?

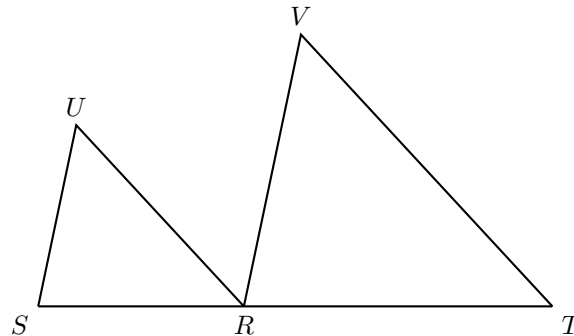
6. Brevet-style questions (French national exam at the end of Year 10)

Questions 1 to 6 are at 🎯 level, questions 7 to 9 at 🏠 level. Questions 1 and 2 deliberately reuse the structure of exercise 2.2: this is practice for the exam format.

6.1. Brevet 1: Similar triangles on a line

★★★☆☆

In the figure below: point R lies on the segment $[ST]$; $SR = 4$; $ST = 10$; $RV = 5,4$ and $TV = 7,2$; the lines (SU) and (RV) are parallel; the lines (RU) and (TV) are parallel.

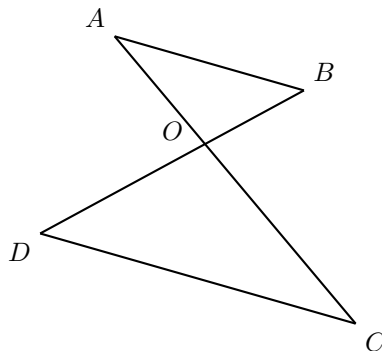


- Prove that the angles $\widehat{USR}$ and $\widehat{VRT}$ have the same size.
- Prove that the triangles SRU and RTV are similar.
- Deduce the lengths SU and RU .

6.2. Brevet 2: The bowtie

★★★☆☆

The segments $[AC]$ and $[BD]$ intersect at O , and the lines (AB) and (CD) are parallel. We are given: $OA = 3$; $OB = 2,4$; $AB = 4,2$; $OC = 5$.



- Prove that the triangles OAB and OCD are similar.
- Calculate CD and OD .

6.3. Brevet 3: Similar triangles and areas

★★★☆☆

The triangles ABC and DEF are similar, with $\hat{A} = \hat{D}$ and $\hat{B} = \hat{E}$. We are given: $AB = 6$ cm; $DE = 4$ cm; the area of triangle ABC is $13,5 \text{ cm}^2$; the perimeter of triangle ABC is 18 cm.

- Give the scale factor from ABC to DEF .
- Calculate the perimeter of triangle DEF .
- Calculate the area of triangle DEF .

6.4. Brevet 4: Recognise, then calculate

★★★☆☆

Triangle JKL satisfies $\hat{J} = 35^\circ$ and $\hat{K} = 60^\circ$. Triangle MNP satisfies $\hat{M} = 85^\circ$ and $\hat{N} = 60^\circ$.

- Prove that these two triangles are similar.
- State the three pairs of corresponding vertices.
- We are given $JK = 8$ cm, $KL \approx 4,6$ cm (rounded to the nearest tenth) and $PN = 6$ cm. Estimate the length NM from this information.

6.5. Brevet 5: The flat floor plan

★★★☆☆

An architect provides the floor plan of a flat at a scale of $\frac{1}{50}$. The ceiling height is 2,50 m.

- On the plan, the living room has a length of 12 cm. What is its actual length, in metres?
- On the plan, the bedroom has an area of 48 cm^2 . What is its actual area, in m^2 ?
- One piece of information in the problem was not needed. Which one, and why is this normal?

6.6. Brevet 6: The 1/18 scale model car

★★★☆☆

(Twin of exercise 4.2, for revision.) A model of a vintage car is at a scale of $\frac{1}{18}$.

- The real car measures 4,32 m in length. What is the length of the model, in cm?
- The real windscreen has an area of $1,296 \text{ m}^2$. What is the area of the model's windscreen, in cm^2 ?
- The real boot has a volume of 291,6 L. What is the volume of the model's boot, in mL?

6.7. Brevet 7: The two trays

★★★★☆

A craftsperson makes two similar triangular trays. The small one has a perimeter of 60 cm and an area of 150 cm^2 ; the large one has a perimeter of 90 cm.

- Justify that the scale factor from the small one to the large one is 1,5. (Why is the ratio of the perimeters equal to k ?)
- Calculate the area of the large tray.
- The craftsperson sells the small tray for 24 € and claims: "the large one costs 1,5 times as much, because it is 1,5 times as big". If the price is proportional to the amount of wood (the area), how much should the large tray cost?

6.8. Brevet 8: Photo print sizes 📷

★★★★

Standard photo prints have the following sizes (in cm): 10×15 , 13×18 and 40×60 .

- Is the 40×60 size an enlargement of the 10×15 size? If so, with what scale factor?
- In that case, what is the area of the print multiplied by?
- Is the 13×18 size an enlargement of the 10×15 size? Explain why the lab has to crop the image if it wants to fill this paper without borders and without distorting the image.

6.8. Optional enrichment 📷

A rectangle that keeps its aspect ratio when cut in half

After photo sizes, we look for a special sheet format. A rectangle is cut into two congruent rectangles, perpendicular to its long sides. We would like each half, after a quarter-turn, to have the same shape as the whole sheet.

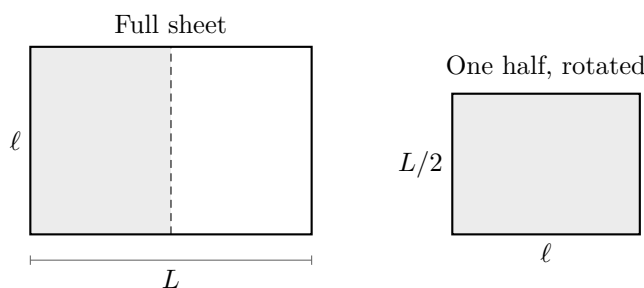
Useful concepts before starting

The long side has length L and the short side ℓ , with $0 < \ell < L$. Both numbers are in the same unit. The aspect ratio is $r = L/\ell$: it compares the long side to the short side. Two rectangles have the same shape precisely when their aspect ratios are equal. Example: 4×6 and 10×15 both have the ratio $3/2$, even if one of the sheets is rotated.

Cutting the long side in half gives halves with dimensions ℓ and $L/2$. They must be ordered from smallest to largest before comparing the formats: the long side of the half is not necessarily the one coming from the original long side.

For $u > 0$, $\sqrt{u}$ is the positive number whose square is u . For example, $\sqrt{9} = 3$, since $3^2 = 9$. If $x > 0$ and $x^2 = u$, we therefore write $x = \sqrt{u}$. A rounded decimal value can help with drawing, but does not replace an exact equality.

A. Understanding the cut and the matching



- Test a sheet of dimensions 10×15 . Give the dimensions of one half, its long side and its short side. Compare its aspect ratio to that of the whole sheet. Is having half the area enough to guarantee the same shape?
- Rule out overly elongated sheets: if $L \geq 2\ell$, the long side of a half is $L/2$ and the short side is ℓ (equal if $L = 2\ell$). Compare the two quotients of corresponding lengths. Why can this case not work?

B. Finding the format and checking that it works

We now look among rectangles such that $\ell < L < 2\ell$. The long side of a half is therefore ℓ and the short side is $L/2$.

- Write the equality between the two “half / whole sheet” scale factors. Show that it requires $L^2 = 2\ell^2$, and then $r^2 = 2$. What is the only possible positive aspect ratio?
- Conversely, check that $L^2 = 2\ell^2$ does indeed give the same scale factors on both sides. For $\ell = 10$ cm, give the exact value of L , then its value rounded to the nearest hundredth. Give the exact reduction scale factor to one half and check its effect on the area.
- Repeat the same cut on one half: why is the format still preserved? After two successive cuts, what fraction of the initial area does one of the four pieces represent? What is the length reduction scale factor compared to the original sheet?

C. Extending and distinguishing exact from approximate

- We now look for a new format, with $\ell < L < 3\ell$, that is preserved when the long side is divided into three equal parts. The three pieces have dimensions ℓ and $L/3$. Determine the required aspect ratio and check that it is sufficient. Give the reduction scale factor and the fraction of the area for one piece.
- A sheet measures exactly 21 cm by 29,7 cm. Does it keep its exact format after being cut in half? Compare $29,7^2$ and 2×21^2 . Explain why a small difference can be invisible on a drawing without being zero.

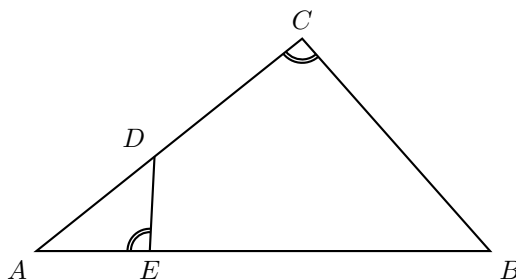
To remember

To recover the same shape after cutting, we first identify the corresponding sides, rotating a piece if needed. We then express shape preservation through equal ratios and check the solution. Equal areas alone are not enough.

6.9. Brevet 9: The hidden common angle 🏰

★★★★

In triangle ABC , E is a point on $[AB]$ and D is a point on $[AC]$ such that $\widehat{AED} = \widehat{ACB}$. We are given: $AE = 3$; $AC = 9$; $AD = 4$; $BC = 7,5$.



- Prove that the triangles AED and ACB are similar.
- State the corresponding sides, then calculate ED and AB .
- Check that E does indeed lie inside the segment $[AB]$ with the length found. Why is this check useful?

7. Going further (sorted bank)

Bank sorted by type: cross-curricular (7.1 to 7.3), extensions (7.4 to 7.6 and 7.8), challenges (7.7, 7.9 and 7.10). Exercises 7.8 and 7.9 prove Pythagoras' theorem in two different ways; 7.10 compares them with a third.

7.1. Geography: reading a map 🗺️

★★★☆☆

- On a map of France at a scale of $\frac{1}{2\,000\,000}$, the distance from Paris to Lyon as the crow flies measures 19,6 cm. What is the actual distance, in km?
- By motorway, the journey is about 465 km. Why do the two answers differ? Which one does the map give?
- On a regional map at a scale of $\frac{1}{200\,000}$, a lake covers 4,5 cm². Calculate its actual area in km² (reuse the powers of 10 from chapter 01: 1 km² = 10¹⁰ cm²).

7.2. Biology: under the microscope 🔬

★★★☆☆

A photograph taken under a microscope is printed so that all lengths of the object are enlarged with scale factor $k = 400$.

- A cell measures 50 μm (1 μm = 0,001 mm). What is its size on the printed photograph, in cm?
- What is the area of the cell's image multiplied by?
- Write 50 μm in metres, in standard form (scientific notation, chapter 01).

7.3. The impossible giant 🏰

★★★★

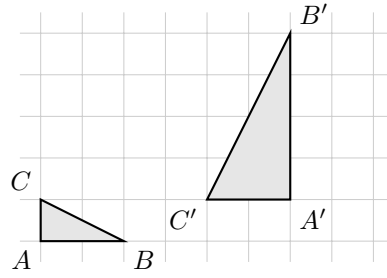
In fairy tales, a giant is a human enlarged with scale factor 10. Let us take a man who is 1,80 m tall and weighs 80 kg.

- How tall would the giant be?
- Assuming the same materials and the same densities: mass is then proportional to body volume. What would the giant's mass be, in tonnes?
- The strength of a bone is proportional to its cross-sectional area (the area of the "slice"). What is this area multiplied by?
- How many times more mass must each cm² of the giant's bone therefore support compared to the man? Explain why the giant would be more fragile. Can we state that his bones would break without knowing their initial strength? (This is the "square-cube law", which also explains why large animals have proportionally thicker legs.)

7.4. Transformations to get from one to the other

★★★☆☆

(Preparation for chapter 06.) On the grid below (squares of side 1), consider the triangles $T_1 = ABC$ and $T_2 = A'B'C'$.



- By reading the lengths on the grid, check that T_2 is similar to T_1 and give the scale factor from T_1 to T_2 .
- Describe a sequence of known transformations (translation, quarter-turn, enlargement) that maps T_1 onto T_2 . Several answers are possible: suggest one, described precisely.

The “enlargement with a fixed centre” used in b) is called a homothety (or enlargement with a centre): it is studied in chapter 06.

7.5. The converse of the characterisation

★★★★

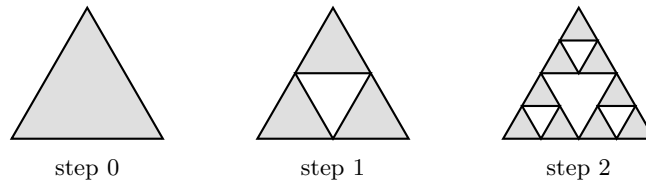
In the course notes, the converse direction is presented as an extension. Let us go through its proof. Let ABC and $A'B'C'$ be two triangles such that $\frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC} = k$, with $0 < k < 1$. The case $k = 1$ is settled by the “side-side-side” congruence criterion; if $k > 1$, we swap the roles of the triangles to obtain a scale factor $1/k < 1$.

- Place M on $[AB]$ such that $AM = A'B'$; the line parallel to (BC) passing through M intersects $[AC]$ at N . Draw a figure.
- Justify that $\frac{AM}{AB} = \frac{AN}{AC} = \frac{MN}{BC}$ (learnt in Year 9).
- Show that $\frac{AM}{AB} = k$, then deduce $AN = A'C'$ and $MN = B'C'$.
- Deduce that the triangles AMN and $A'B'C'$ are congruent (state the congruence criterion), and then that the angles of $A'B'C'$ are equal to those of ABC (carefully justify the step from AMN to ABC : which angles are equal, and why?).

7.6. The Sierpiński triangle

★★★★

(Looking ahead to upper secondary school.) We start with an equilateral triangle of side 8 cm, with area denoted by $\mathcal{A}$ (step 0). At each step, each triangle is replaced by the 3 “corner” triangles, obtained by joining the midpoints of the sides (the central triangle is removed).



- Justify that each small triangle at step 1 is a reduction with scale factor $\frac{1}{2}$ of the initial triangle. What is the area of each one, in terms of $\mathcal{A}$?
- Deduce the total shaded area at step 1, then at step 2, in terms of $\mathcal{A}$. Write it in the form $\left(\frac{3}{4}\right)^n \mathcal{A}$ (powers from chapter 01!).
- Calculate the sum of the perimeters of the shaded triangles at steps 0, 1 and 2. What do you notice?
- If we continue indefinitely, what does the shaded area seem to approach? What about the total perimeter? (Answer in one sentence: this is the paradox of fractals.)

7.7. Challenge: almost twins

★★★☆☆

Short problem, pure reasoning: everyone can have a go.

Do there exist two triangles that are similar but not congruent and share two side lengths? Find an example (with explicit lengths, and a check that they do indeed form triangles) or prove that it is impossible.

7.7. Optional enrichment

Almost twins: from one example to a whole family

To be read after attempting the initial challenge.

The initial exercise asks for one example. We now look for all pairs of similar triangles with scale factor $3/2$, having two distinct side lengths in common, with all six lengths being integers in centimetres. Triangles are described by their side lengths: moving or flipping a figure does not create a new solution.

Useful concepts before starting

A common length appears in both triangles. This does not mean that the sides of this length are corresponding sides. Corresponding sides occupy the same rank when the side lengths of each triangle are ordered from smallest to largest.

Example already found: $(4; 6; 9)$ and $(6; 9; 13,5)$ are similar with scale factor $3/2$. Their common lengths are 6 and 9, but the side of length 6 in the first corresponds to the side of length 9 in the second.

Two distinct lengths mean two different values: two sides each measuring 6 cm only represent one distinct length. For three numbers $0 < a \leq b \leq c$, a non-degenerate triangle exists if and only if $a + b > c$.

Arithmetic reminder: a product of two odd integers is odd. Therefore, if $3u$ or $9u$ is even, u is even. For example, 3×5 is odd, whereas 3×6 is even. A positive even integer can be written as $2v$, where v is a positive integer.

A. Understanding how two lengths can match

- In this part, we consider an enlargement with scale factor $k > 1$.
- A1) In the example $(4; 6; 9)$ and $(6; 9; 13,5)$, write the three quotients of corresponding sides. State the rank of the length 6 in each triangle, then do the same for 9. Why does this example not yet satisfy the new integer-length constraint?
- A2) An isosceles triangle, possibly equilateral, has at most two distinct side lengths. Explain why its smallest length does not appear in an enlargement with scale factor $k > 1$. Can it then share two distinct lengths with this enlargement?
- A3) We can now denote by $a < b < c$ the three lengths of the first triangle. Those of the second are $ka < kb < kc$. Why is a not common to both triangles? Why is kc not common? Deduce that the two common lengths are necessarily b, c in the first and ka, kb in the second. Then justify $b = ka$ and $c = kb$: here we match the common values in order, not the corresponding sides.

B. Imposing the scale factor $3/2$ and integer lengths

We now set $k = 3/2$ and keep the notation a for the shortest side of the first triangle.

- B1) Using A3, complete:

	shortest side	middle side	longest side
first triangle	a	$\dots$	$\dots$
second triangle	$\dots$	$\dots$	$\frac{27a}{8}$

- For any $a > 0$, check that the sum of the two shortest sides of the first triangle is greater than the third. Why is the second one then also a triangle?
- B2) All six lengths must be integers. Why does $3a/2$ being an integer require $a = 2u$ for some integer u ? Substituting into $9a/4$, then show that $u = 2v$. Finally, substituting into $27a/8$, show that $v = 2n$. Deduce that a is a multiple of 8.
- B3) Replace a with $8n$ in the six lengths. Conversely, check that every integer $n \geq 1$ does indeed provide two non-congruent triangles, similar with scale factor $3/2$, with integer side lengths and two distinct lengths in common. Conclude: have we found just a few examples or all the solutions?
- B4) What choice makes the shortest side a as small as possible? Justify both that it works and that no smaller choice is possible.

C. Adding a bound, then changing a hypothesis

- C1) We additionally require that the longest side of the second triangle measure at most 100 cm. Find all possible pairs of triangles and give their six side lengths in a table. Justify where the list stops.
- C2) We return to arbitrary positive lengths, but replace the scale factor $3/2$ with 2. Can two non-congruent similar triangles then have two distinct side lengths in common? Repeat A3, then check the triangle inequality. Is a sequence of proportional numbers enough to form triangles?

To remember

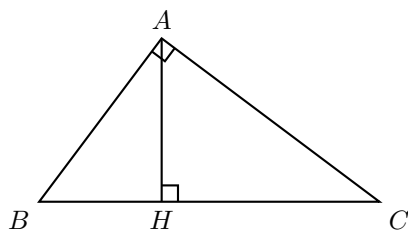
To describe all solutions, we first show the form that any solution must take, and then check that every object of this form works. Arithmetic constraints do not replace the geometric condition: the lengths must actually form triangles.

7.8. Metric relations in a right-angled triangle: Pythagoras proved again

★★★★

This exercise reuses the angle characterisation from the course notes: Pythagoras' theorem is proved here, not assumed.

ABC is a triangle right-angled at A , and H is the foot of the altitude from A : $H \in [BC]$ and $(AH) \perp (BC)$.



- Prove that the triangles ABC and HBA are similar, and carefully justify the matching $\frac{BC}{BA} = \frac{CA}{AH} = \frac{AB}{HB}$ (which angles correspond to which?).
- Deduce that $AH = \frac{AB \times AC}{BC}$, and then that $AB^2 = BH \times BC$ (Euclid's theorem).
- Similarly, prove that the triangles ABC and HAC are similar, and deduce that $AC^2 = CH \times CB$.
- Using the ratios from a) and c), establish $AH = HB \times \frac{CA}{AB}$ and then $AH = CH \times \frac{AB}{CA}$, and deduce that $AH^2 = HB \times HC$ (the altitude relation, sometimes called "Descartes' relation").
- By adding the two equalities from Euclid's theorem (questions b and c), prove that $AB^2 + AC^2 = BC^2$: this is Pythagoras' theorem.
- Numerical check. In the triangle with $AB = 3$, $AC = 4$, $BC = 5$: calculate AH , BH and CH using the formulas, then check the relations from b), c) and d), and check that $BH + CH = BC$.

7.8. Optional enrichment

Constructing a square root with a semicircle

To be tackled after establishing the relation $AH^2 = BH \times HC$ in exercise 7.8. We are going to read it the other way round: choosing BH and HC to construct a length AH whose square is known.

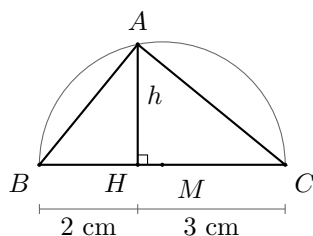
Useful concepts before starting

For $u > 0$, $\sqrt{u}$ is the positive number whose square is u . Example: $\sqrt{9} = 3$, because $3^2 = 9$. If a positive length x , measured in centimetres, satisfies $x^2 = 6$, we therefore know that it is equal to $\sqrt{6}$ cm, even without having an exact decimal expansion.

We use the circle property: if A lies on a circle with diameter $[BC]$, with $A \neq B, C$, then ABC is right-angled at A . Constructing a point on this circle therefore provides the right angle needed for the altitude relation.

An exact construction is justified by the properties of the objects: midpoint, circle, perpendicularity, equal lengths. The physical drawing only has the accuracy of the instruments; measuring it alone does not constitute a proof. In the calculations, the lowercase letters p, q, h will denote the numerical values of the lengths in centimetres.

A. Constructing a length whose square is 6



- Carry out this construction: draw a segment $[BC]$ of length 5 cm; place $H \in [BC]$ with $BH = 2$ cm; mark the midpoint M of $[BC]$; draw the semicircle with centre M , above (BC) ; the perpendicular to (BC) at H meets this semicircle at A . What radius should be used? What is the length of HC ?
- Justify that ABC is right-angled at A , and then that $AH^2 = 6 \text{ cm}^2$. Deduce the exact value of AH and check that it lies strictly between 2 and 3 cm.
- Measuring gives about 2,45 cm. Calculate $2,45^2$: is this decimal value the exact length? Which step of the construction proves the exact value without relying on the markings on a ruler?

B. Understanding the geometric machine

- B1) Repeat the construction with $BH = p$ cm and $HC = q$ cm, where $p, q > 0$. Give BC , the radius of the semicircle, and then the length AH in terms of p and q . Explain why the justification works for all such choices.
- B2) To obtain $AH = \sqrt{6}$ cm again, we now choose $BH = 1$ cm and $HC = 6$ cm. Justify this result. Are the two large triangles constructed in this way, with hypotenuses of 5 and 7 cm, similar? Use the fact that a similarity also multiplies the altitudes by its scale factor.

C. Choosing the given lengths, then finding the maximum

- C1) Construct a segment of length $\sqrt{7}$ cm using this method. Choose the two lengths along the diameter, give the radius to use and justify the result. Only one valid choice is required.
- C2) We now fix $BC = 10$ cm, but let H move strictly between B and C . What is the greatest possible altitude AH and where should H be placed? Hint: let M denote the centre of the semicircle. Handle the case $H = M$ directly, then compare AH with the radius AM when $H \neq M$, using Pythagoras' theorem in AHM .

To remember

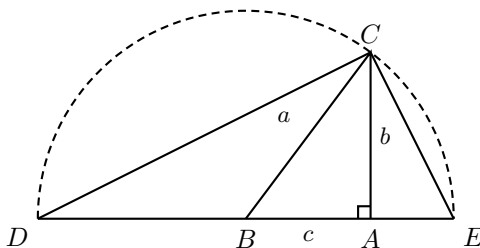
A geometric formula can also be used to design a construction. The semicircle guarantees the right angle; similar triangles link the square of the altitude to the product of the two segments of the diameter. Choosing the given lengths, constructing and proving are three distinct steps.

7.9. Challenge: Pythagoras using a circle 🏰

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There are hundreds of proofs of Pythagoras' theorem. Here is one that uses a circle and similar triangles: three questions are enough.

ABC is a triangle right-angled at A . We set $AB = c$, $AC = b$ and $BC = a$. The circle with centre B passing through C intersects the line (AB) at two points: E on the side of A , and D on the other side, so the points lie in the order D, B, A, E .



- a) Justify that $AE = a - c$ and that $AD = a + c$. (By the way: why can we be sure that $c < a$?)
- b) Prove that triangle DEC is right-angled at C . (Year 9 reminder: if a triangle is inscribed in a circle and one of its sides is a diameter, then it is right-angled.)
- c) Prove that the triangles CAE and DAC are similar, deduce that $\frac{a-c}{b} = \frac{b}{a+c}$, and then that $a^2 = b^2 + c^2$.

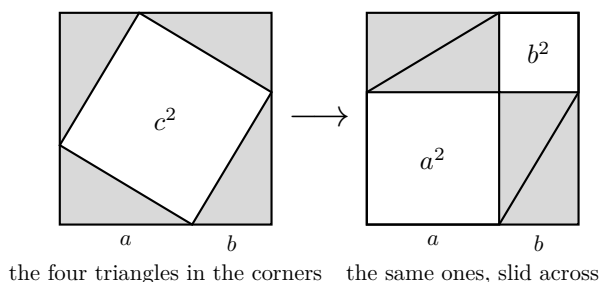
For the curious: three other proofs “by similar triangles” exist along the same lines: one using the circle with centre B passing through A , another that puts three similar triangles together into a rectangle, and a third using an angle bisector. Plenty of material for a presentation.

7.10. Two proofs of Pythagoras: which one gives more insight? 🧩

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(Pythagoras' theorem was learnt in Year 9: here we are not (re)discovering the theorem, we are comparing two ways of proving it. To be done after 7.8.)

Here is a second proof of Pythagoras' theorem, "by dissection". We take a large square of side $a + b$ and four copies of the same right-angled triangle, whose legs have lengths a and b , and whose hypotenuse has length c .



- In the left-hand figure, justify that the central white region has all four sides equal to c (they are hypotenuses) and four right angles. (Hint: at each vertex, the two acute angles of a right-angled triangle are complementary, so their sum is 90° .) Deduce that its area is c^2 .
- In the right-hand figure, the same four triangles have been slid across. Justify that the two white regions are two squares, with areas a^2 and b^2 .
- In both cases, we have the same large square and the same four triangles: the remaining white region therefore has the same area on both sides. Deduce that $c^2 = a^2 + b^2$.
- Cut out the puzzle from paper (one large square and four identical right-angled triangles) and experience "hands-on" how to move from one figure to the other.
- Comparing two proofs. In 7.8, we proved Pythagoras' theorem in another way, using similar triangles. Of the two proofs, which one convinces you more quickly? Which one makes it clearer why the theorem is true? Write your opinion in a few lines, with an argument.

End of the exercise sheet: full worked solutions in the document "Worked solutions to the exercise sheet".